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THESIS

INFINITE IMPULSE RESPONSE
NOTCH FILTER

by

Venus Jangsri

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Thesis Advisor: H. H. Loomis, Jr.

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A pipeline technique by Loomis and Sinha has been applied to the design of recursive digital filters. Recursive digital filters operating at hitherto impossibly high rates can be designed by this technique.

An alternate technique by R. Gnanasekaran allows high speed implementation using the state-space structure directly. High throughput is also achieved by use of pipelined multiply-add modules. The actual hardware complexity will depend upon the number of pipeline stages.

These techniques are used for the design of the IIR notch filter and finally, a comparison of the performance and complexity of these two techniques is presented.

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Infinite Impulse Response Notch Filter

by

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ABSTRACT

A pipeline technique by Loomis and Sinha has been applied to the design of recursive digital filters. Recursive digital filters operating at hitherto impossibly high rates can be designed by this technique.

An alternate technique by R. Gnanasekaran allows high speed implementation using the state-space structure directly. High throughput is also achieved by use of pipelined multiply-add modules. The actual hardware complexity will depend upon the number of pipeline stages.

These techniques are used for the design of the IIR notch filter and finally, a comparison of the performance and complexity of these two techniques is presented.

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I. INTRODUCTION

A. DIRECT FORM STRUCTURE REALIZATION

A scheme for high speed recursive digital filter realization was suggested by H. H. Loomis and B. Sinha [Ref. 1]. The main aspect of the design is in modifying the filter difference equation so that the recursive part requires only the Lth past output instead of the immediate past output. This removes the tight recursive requirement thus enabling pipelining of multipliers and adders in the direct form implementation resulting in high throughput. One of the difficulties with this method is that the modification introduces extra poles (not necessarily stable) and zeros in the filter transfer function which cancel each other in the infinite arithmetic implementation. In practice due to the finite word length effect this cancellation may not occur. This may result in an unstable realization even if the original filter were a stable one. However, if L is chosen to be large enough, then it has been proven that the new poles introduced will also be within the unit circle thus guaranteeing stability even if the pole-zero cancellation does not occur. This method is called the direct form structure, or the scalar structure.

B. STATE SPACE REALIZATION

Another method by R. Gnanasekaran is an alternate technique which relaxes the recursive nature of the state equation (state space realization) [Ref. 2]. That is, the present state vector computation does not require the immediate past state vector. The proposed technique allows high speed implementation using the state-space structure directly. High throughput is achieved by use of pipelined multiply -add modules. The actual hardware complexity will depend upon the method of pipelining, though very high speed is possible by increasing the number of states in the pipelining. This method implements $\mathbf{A}^L$ instead of $\mathbf{A}$, the system matrix. The statespace realization will be stable if the original filter is stable. Furthermore, the $\mathbf{A}^L$ filter has improved tolerance to coefficient error.

C. OVERVIEW OF THESIS

Chapter II describes the IIR notch filter. The transfer function and the coefficients that are appropriate for the IIR notch filter are developed.

Chapter III presents the realization of the high speed IIR notch filter by using the pipeline technique of Loomis and Sinha. Stability testing, frequency response and unit

sample response are described. System designs for pipelined multiply-add modules are included.

Chapter IV presents Gnanasekaran's state space realization which includes a modified state space equation. System designs for pipelined multiply-add modules are described.

Chapter V contains a comparison of the results of the Loomis/Sinha pipeline recursive design and the Gnanasekaran state space realization design.

Appendix A contains the program and data for frequency response analysis of the IIR notch filters in Chapter III.

Appendix B contains the program and data for unit sample response analysis of the IIR notch filters in Chapter III.

II. IIR NOTCH FILTER DESIGN

From Rowe, [Ref. 3: p. 72] the IIR notch filter has the transfer function

$$H(z) = \frac{b_0 z^2 + b_1 z + b_0}{a_0 z^2 + b_1 z + a_2}. \tag{2.1}$$

Dividing the numerator and denominator of (2.1) by a_0, we have

$$H(z) = \frac{\frac{b_0}{a_0} z^2 + \frac{b_1}{a_0} z + \frac{b_0}{a_0}}{z^2 + \frac{b_1}{a_0} z + \frac{a_2}{a_0}}. \tag{2.2}$$

Let $\frac{b_0}{a_0} = B_0$, $\frac{b_1}{a_0} = B_1$, and $\frac{a_2}{a_0} = A_0$. Therefore the transfer function is

$$H(z) = \frac{B_0 z^2 + B_1 z + B_0}{z^2 + B_1 z + A_0}. \tag{2.3}$$

From (2.1) and (2.3) we can see that

1. B_1 in the numerator, must be equal to B_1 in the denominator; and,
2. the first B_0 in the numerator must be equal to the last B_0 in the numerator.

Rewriting (2.3) as

$$H(z) = \frac{B_0(z^2 + \frac{B_1}{B_0} z + 1)}{(z^2 + B_1 z + A_0)}, \tag{2.4}$$

the transfer function $H(z)$ is of the form

$$H(z) = \frac{B_0(z - n_1)(z - n_2)}{(z - d_1)(z - d_2)}. \tag{2.5}$$

Let the zeros be placed at $1e^{\pm j\theta}$, and let the poles be placed at $\sqrt{A_0}\, e^{\pm j\theta}$. Therefore the transfer function is

$$H(z) = \frac{B_0(z - 1e^{-j\theta})(z - 1e^{+j\theta})}{(z - \sqrt{A_0}\,e^{-j\theta})(z - \sqrt{A_0}\,e^{+j\theta})} \tag{2.6}$$

$$H(z) = \frac{B_0[z^2 - 1e^{+j\theta}z - 1e^{-j\theta}z + 1]}{[z^2 - \sqrt{A_0}\,e^{+j\theta}z - \sqrt{A_0}\,e^{-j\theta}z + \sqrt{A_0}\,]} \tag{2.7}$$

$$H(z) = \frac{B_0[z^2 - 2(\dfrac{e^{+j\theta} + e^{-j\theta}}{2})z + 1]}{[z^2 - 2\sqrt{A_0}\,(\dfrac{e^{+j\theta} + e^{-j\theta}}{2})z + A_0]} \tag{2.8}$$

$$H(z) = \frac{B_0[z^2 - 2\cos\theta z + 1]}{[z^2 - 2\sqrt{A_0}\,\cos\theta z + A_0]} \tag{2.9}$$

$$H(z) = \frac{B_0 z^2 - 2B_0\cos\theta z + B_0}{z^2 - 2\sqrt{A_0}\,\cos\theta z + A_0}\,. \tag{2.10}$$

According to (2.3) we can see that

$$-2B_0\cos\theta z = -2\sqrt{A_0}\,\cos\theta z \tag{2.11}$$

which means

$$B_0 = \sqrt{A_0}\,. \tag{2.12}$$

For our example of the IIR notch filter we want the zeros to be placed at

$$1e^{\pm j0.4467110339\pi} = 1e^{\pm j\theta}. \tag{2.13}$$

and we want the poles to be placed at

$$0.99e^{\pm j0.4467110339\pi} = \sqrt{A_0}\,e^{\pm j\theta}. \tag{2.14}$$

We can see that

$$\theta = 0.4467110339\pi \tag{2.15}$$

$$A_0 = 0.9801 \tag{2.16}$$

$$B_0 = \sqrt{A_0} = 0.99\,. \tag{2.17}$$

This realization is shown in Figure 1 on page 6.

From equation (2.1)

$$H(z) = \frac{B_0 z^2 - 2B_0 \cos\theta z + B_0}{z^2 - 2\sqrt{A_0}\ \cos\theta z + A_0}\,. \tag{2.18}$$

Substituting θ, B_0 and A_0, we then have

$$H(z) = \frac{0.99z^2 - 2(0.99)\cos(0.4467110339\pi)z + 0.99}{z^2 - 2(0.99)\cos(0.4467110339\pi)z + 0.9801} \tag{2.19}$$

$$H(z) = \frac{0.99z^2 - 0.32993z + 0.99}{z^2 - 0.32993z + 0.9801} \tag{2.20}$$

$$H(z^{-1}) = \frac{0.99 - 0.32993z^{-1} + 0.99z^{-2}}{1 - 0.32993z^{-1} + 0.9801z^{-2}}\,. \tag{2.21}$$

Because we have the transfer function [Ref. 1: p. 273]

$$H(z^{-1}) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 - b_1 z^{-1} - b_2 z^{-2}}\,, \tag{2.22}$$

from (2.21) and (2.22) we can see that all of coefficients are

$$\begin{aligned} a_0 &= 0.99 \\ a_1 &= -0.32993 \\ a_2 &= 0.99 \\ b_1 &= 0.32993 \\ b_2 &= -0.9801\,. \end{aligned} \tag{2.23}$$

$$H(z^{-1}) = \frac{Y(z)}{X(z)} \tag{2.24}$$

$$\frac{Y(z)}{X(z)} = \frac{0.99 - 0.32993z^{-1} + 0.99z^{-2}}{1 - 0.32993z^{-1} + 0.9801z^{-2}} \tag{2.25}$$

$$Y(z)[1 - 0.32993z^{-1} + 0.9801z^{-2}] = X(z)[0.99 - 0.32993z^{-1} + 0.99z^{-2}] \tag{2.26}$$

$$Y(z) - 0.32993z^{-1}Y(z) + 0.9801z^{-2}Y(z) = 0.99X(z) - 0.32993z^{-1}X(z) + 0.99z^{-2}X(z) \tag{2.27}$$

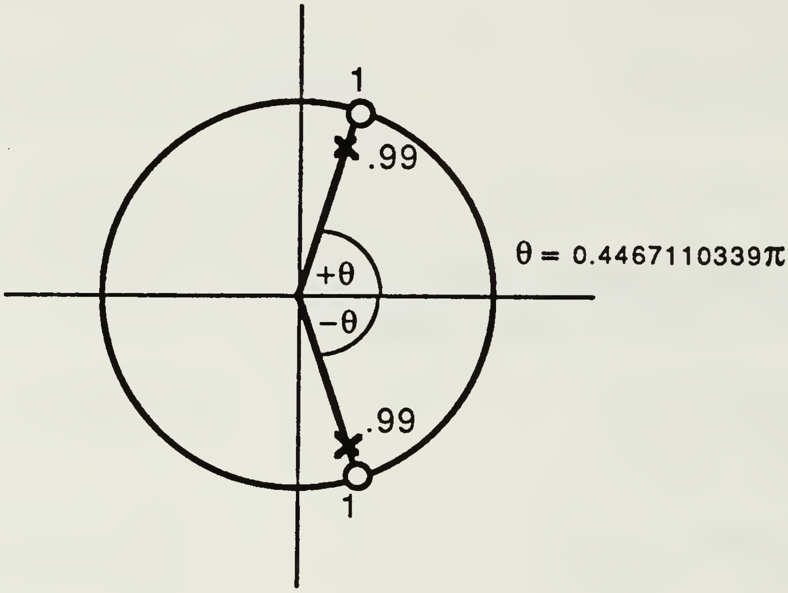

Figure 1. Poles and Zeros Location

$$Y(z) = 0.99X(z) - 0.32993z^{-1}X(z) + 0.99z^{-2}X(z) + 0.32993z^{-1}Y(z) - 0.9801z^{-2}Y(z) \tag{2.28}$$

$$y(n) = 0.99x(n) - 0.32993x(n-1) + 0.99x(n-2) + 0.32993y(n-1) - 0.9801y(n-2). \tag{2.29}$$

This is the basic second-order recursive filter (p = 0). The number of p will be explained later in Chapter III and the realization is shown in Figure 2 on page 7.

We can design this IIR notch filter to be a high speed IIR notch filter by using pipeline techniques so that it can be operated at hithero impossibly high rates. In Chapter III we will apply Loomis/Sinha method and in Chapter IV Gnanasekaran's state space realization. A comparison of the results of these two techniques will be shown in Chapter V.

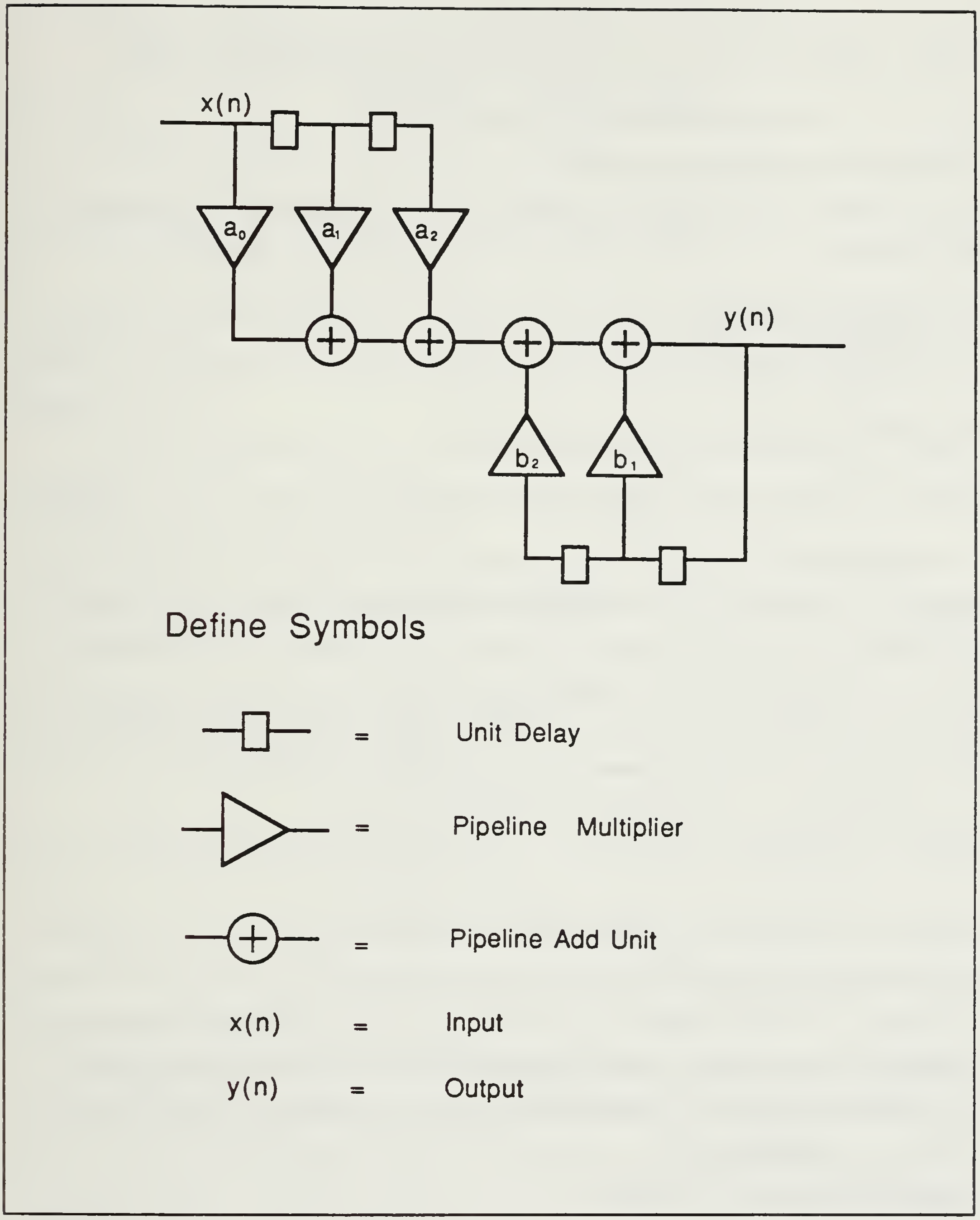

Figure 2. Second Order IIR Notch Filter: $p = 0$

III. REALIZATION OF IIR NOTCH FILTER BY USING THE LOOMIS/SINHA METHOD

A. PIPELINE RECURSIVE DESIGN

From Chapter II we have the second order IIR notch filter that is described by the transfer function

$$H(z^{-1}) = \frac{(a_0 + a_1 z^{-1} + a_2 z^{-2})}{(1 - b_1 z^{-1} - b_2 z^{-2})} \tag{3.1}$$

and the difference equation

$$y(n) = b_1 y(n-1) + b_2 y(n-2) + a_0 x(n) + a_1 x(n-1) + a_2 x(n-2). \tag{3.2}$$

The computation of $y(n)$ requires the immediate past output $y(n-1)$. This filter can be developed into high speed IIR notch filter by using the pipeline technique of Loomis and Sinha [Ref. 1]. The main aspect of the design is in modifying this filter difference equation so that the recursive part requires only from the **Lth** past output instead of the immediate past output. This means the speed limits can be overcome by modifying this difference equation so that $y(n)$ depends not on the immediate past output, $y(n-1)$, but on $y(n-L)$ for some $L > 1$. This can be accomplished as follows:

For $L = 1$ the difference equation is

$$y(n) = b_1 y(n-1) + b_2 y(n-2) + a_0 x(n) + + a_1 x(n-1) + a_2 x(n-2). \tag{3.3}$$

We can see that this equation is a 2-order difference equation.

From [Ref. 1: p. 274] the difference equation has $(p+m)^{th}$-order where m is the order of the difference equation and p is the pipeline delay used to overcome the speed limit. Therefore equation (3.3) has the number of p = 0. We can call this equation the second order IIR notch filter with p = 0.

Delaying equation (3.3) by 1 yields

$$y(n-1) = b_1 y(n-2) + b_2 y(n-3) + a_0 x(n-1) + a_1 x(n-2) + a_2 x(n-3). \tag{3.4}$$

For $L = 2$, putting (3.4) into (3.3) results in the difference equation

$$y(n) = b_1[b_1y(n-2) + b_2y(n-3) + a_0x(n-1) + a_1x(n-2) + a_2x(n-3)] + b_2y(n-2) + a_0x(n) + a_1x(n-1) + a_2x(n-2) \tag{3.5}$$

$$y(n) = b_1^2y(n-2) + b_1b_2y(n-3) + a_0b_1x(n-1) + a_1b_1x(n-2) + a_2b_1x(n-3) + b_2y(n-2) + a_0x(n) + a_1x(n-1) + a_2x(n-2) \tag{3.6}$$

$$y(n) = (b_1^2 + b_2)y(n-2) + b_1b_2y(n-3) + a_0x(n) + (a_0b_1 + a_1)x(n-1) + (a_1b_1 + a_2)x(n-2) + a_2b_1x(n-3). \tag{3.7}$$

We see that the computation of $y(n)$ requires only the second past output $y(n-2)$ instead of immediate past output $y(n-1)$, and we can use this technique for the higher-order of L .

We see that equation (3.7) is a 3-order difference equation and has p = 1. Thus we can call this filter the augmented IIR notch filter for p = 1.

The z transform of equation (3.7) is

$$Y = (b_1^2 + b_2)z^{-2}Y + b_1b_2z^{-3}Y + a_0X + (a_0b_1 + a_1)z^{-1}X + (a_1b_1 + a_2)z^{-2}X + a_2b_1z^{-3}X \tag{3.8}$$

$$\frac{Y}{X} = \frac{a_0 + (a_0b_1 + a_1)z^{-1} + (a_1b_1 + a_2)z^{-2} + a_2b_1z^{-3}}{1 - (b_1^2 + b_2)z^{-2} - b_1b_2z^{-3}} \tag{3.9}$$

$$H'(z^{-1}) = \frac{(a_0 + a_1z^{-1} + a_2z^{-2})(1 + b_1z^{-1})}{(1 - b_1z^{-1} - b_2z^{-2})(1 + b_1z^{-1})}\,. \tag{3.10}$$

That means the transfer function of the augmented IIR notch filter, p = 1, is equal to the transfer function of the second order IIR notch filter, p = 0, multiplied and divided by the same polynomials.

We can design the higher-order difference equation by starting from a second-order difference equation. That means we can design the higher order, p-augmented IIR notch filter by starting from our second order IIR notch filter, p = 0, as follows:

From Chapter II the second order IIR notch filter transfer function is the original transfer function

$$H(z^{-1}) = \frac{(a_0 + a_1z^{-1} + a_2z^{-2})}{(1 - b_1z^{-1} - b_2z^{-2})}\,. \tag{3.11}$$

Multiply the numerator and denominator of the original transfer function by the same polynomial equation. This can be thought of as equation (3.12) As developed in [Ref. 1: p. 281], the transfer function of the p-augmented filter is as shown in equation (3.13) We know from other considerations that the transfer function of the higher order realization must in fact be the same as the original and therefore must have the same poles and zeroes in the z - plane.

The coefficients of the α polynomial depend on the original b_1 and b_2, and p is determined by the number of stages in the pipeline multiply and add units.

$$H'(z^{-1}) = \frac{(a_0 + a_1 z^{-1} + a_2 z^{-2})(1 + \alpha_1^{(p)} z^{-1} + \cdots + \alpha_p^{(p)} z^{-p})}{(1 - b_1 z^{-1} - b_2 z^{-2})(1 + \alpha_1^{(p)} z^{-1} + \cdots + \alpha_p^{(p)} z^{-p})} \tag{3.12}$$

$$H'(z^{-1}) = \frac{a_0^{(p)} z^{-1} + \cdots + a_{p+2}^{(p)} z^{-p-2}}{1 - b_{p+1}^{(p)} z^{-p-1} - b_{p+2}^{(p)} z^{-p-2}} \tag{3.13}$$

The values of $b_j^{(p)}$, $\alpha_j^{(p)}$, and $a_j^{(p)}$ are given in Tables 1, 2 and 3 [Ref. 1: pp. 278-279].

Table 1. b's for $p = 0,1,\ldots,5$

	Coefficient of D^{p-1}	Coefficient of D^{p-2}
p	$b_{p-1}^{(p)}$	$b_{p-2}^{(p)}$
0	b_1	b_2
1	$b_1^2 + b_2$	$b_1 b_2$
2	$b_1^3 + 2b_1 b_2$	$b_1^2 b_2 + b_2^2$
3	$b_1^4 + 3b_1^2 b_2 + b_2^2$	$b_1^3 b_2 + 2b_1 b_2^2$
4	$b_1^5 + 4b_1^3 b_2 + 3b_1 b_2^2$	$b_1^4 b_2 + 3b_1^2 b_2^2 + b_2^3$
5	$b_1^6 + 5b_1^4 b_2 + 6b_1^2 b_2^2 + b_2^3$	$b_1^5 b_2 + 4b_1^3 b_2^2 + 3b_1 b_2^3$

Table 2. α 's for $p = 0,1,\ldots,5$

p	$\alpha_0^{(p)}$	$\alpha_1^{(p)}$	$\alpha_2^{(p)}$	$\alpha_3^{(p)}$	$\alpha_4^{(p)}$	$\alpha_5^{(p)}$
0	1	0	0	0	0	0
1	1	b_1	0	0	0	0
2	1	b_1	$b_1^2 + b_2$	0	0	0
3	1	b_1	$b_1^2 + b_2$	$b_1^3 + 2b_1b_2$	0	0
4	1	b_1	$b_1^2 + b_2$	$b_1^3 + 2b_1b_2$	$b_1^4 + 3b_1^2b_2 + b_2^2$	0
5	1	b_1	$b_1^2 + b_2$	$b_1^3 + 2b_1b_2$	$b_1^4 + 3b_1^2b_2 + b_2^2$	$b_1^5 + 4b_1^3b_2 + 3b_1b_2^2$

Table 3. a 's for $p = 0,1,\ldots,3$

p	$a_0^{(p)}$	$a_1^{(p)}$	$a_2^{(p)}$	$a_3^{(p)}$	$a_4^{(p)}$	$a_5^{(p)}$
0	a_0	a_1	a_2	0	0	0
1	a_0	$a_0b_1 + a_1$	$a_1b_1 + a_2$	a_2b_1	0	0
2	a_0	$a_0b_1 + a_1$	$a_0(b_1^2 + b_2)$ $+ a_1b_1 + a_2$	$a_1(b_1^2 + b_2)$ $+ a_2b_1$	$a_2(b_1^2 + b_2)$	0
3	a_0	$a_0b_1 + a_1$	$a_0(b_1^2 + b_2)$ $+ a_1b_1 + a_2$	$a_0(b_1^3 + 2b_1b_2)$ $+ a_1(b_1^2 + b_2)$ $+ a_2b_1$	$+ a_1(b_1^3 + 2b_1b_2)$ $+ a_2(b_1^2 + b_2)$	$+ a_2(b_1^3 + 2b_1b_2)$

B. SECOND ORDER IIR NOTCH FILTER: P = 0

From equation (3.13) in section A, we let p = 0. This equation becomes:

$$H'(z^{-1}) = \frac{(a_0 + a_1z^{-1} + a_2z^{-2})}{(1 - b_1z^{-1} - b_2z^{-2})} \tag{3.14}$$

$$H'(z^{-1}) = \frac{Y(z)}{X(z)} \tag{3.15}$$

$$\frac{Y(z)}{X(z)} = \frac{(a_0 + a_1z^{-1} + a_2z^{-2})}{(1 - b_1z^{-1} - b_2z^{-2})} \tag{3.16}$$

Manipulating (3.16) will result in the difference equation, (3.20).

$$Y(z)[1 - b_1z^{-1} - b_2z^{-2}] = X(z)[a_0 + a_1z^{-1} + a_2z^{-2}] \tag{3.17}$$

$$Y(z) - b_1 z^{-1} Y(z) - b_2 z^{-2} Y(z) = a_0 X(z) + a_1 z^{-1} X(z) + a_2 z^{-2} X(z) \tag{3.18}$$

$$Y(z) = a_0 X(z) + a_1 z^{-1} X(z) + a_2 z^{-2} X(z) + b_1 z^{-1} Y(z) + b_2 z^{-2} Y(z) \tag{3.19}$$

$$y(n) = a_0 x(n) + a_1 x(n-1) + a_2 x(n-2) + b_1 y(n-1) + b_2 y(n-2). \tag{3.20}$$

This realization is shown in Figure 3 on page 13.

From chapter II we know that for our example notch filter, that the coefficient values are

$$\begin{aligned} a_0 &= 0.99 \\ a_1 &= -0.32993 \\ a_2 &= 0.99 \\ b_1 &= 0.32993 \\ b_2 &= -0.9801\ . \end{aligned} \tag{3.21}$$

The transfer function is

$$H'(z^{-1}) = \frac{(a_0 + a_1 z^{-1} + a_2 z^{-2})}{(1 - b_1 z^{-1} - b_2 z^{-2})}\ . \tag{3.22}$$

The difference equation is

$$y(n) = a_0 x(n) + a_1 x(n-1) + a_2 x(n-2) + b_1 y(n-1) + b_2 y(n-2). \tag{3.23}$$

C. AUGMENTED IIR NOTCH FILTER: P = 1

From equations (3.12) and (3.13) in section A, we let p = 1. Those equations become:

$$H'(z^{-1}) = \frac{(a_0 + a_1 z^{-1} + a_2 z^{-2})(1 + \alpha_1^{(1)} z^{-1})}{(1 - b_1 z^{-1} - b_2 z^{-2})(1 + \alpha_1^{(1)} z^{-1})} \tag{3.24}$$

$$H'(z^{-1}) = \frac{a_0^{(1)} + a_1^{(1)} z^{-1} + a_2^{(1)} z^{-2} + a_3^{(1)} z^{-3}}{1 - b_2^{(1)} z^{-2} - b_3^{(1)} z^{-3}} \tag{3.25}$$

$$H'(z^{-1}) = \frac{Y(z)}{X(z)} \tag{3.26}$$

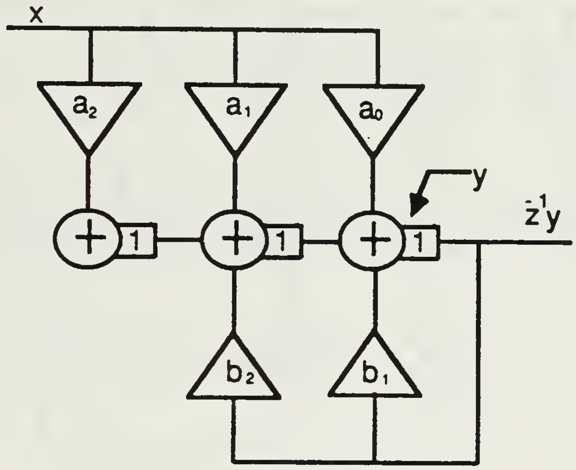

Figure 3. Second Order IIR Notch Filter: p = 0

$$\frac{Y(z)}{X(z)} = \frac{a_0^{(1)} + a_1^{(1)}z^{-1} + a_2^{(1)}z^{-2} + a_3^{(1)}z^{-3}}{1 - b_2^{(1)}z^{-2} - b_3^{(1)}z^{-3}} \tag{3.27}$$

Manipulating (3.27) will result in the difference equation, (3.31).

$$Y(z)[1 - b_2^{(1)}z^{-2} - b_3^{(1)}z^{-3}] = X(z)[a_0^{(1)} + a_1^{(1)}z^{-1} + a_2^{(1)}z^{-2} + a_3^{(1)}z^{-3}] \tag{3.28}$$

$$\begin{aligned} Y(z) - b_2^{(1)}z^{-2}Y(z) - b_3^{(1)}z^{-3}Y(z) = a_0^{(1)}X(z) + a_1^{(1)}z^{-1}X(z) + a_2^{(1)}z^{-2}X(z) \\ + a_3^{(1)}z^{-3}X(z) \end{aligned} \tag{3.29}$$

$$\begin{aligned} Y(z) = a_0^{(1)}X(z) + a_1^{(1)}z^{-1}X(z) + a_2^{(1)}z^{-2}X(z) + a_3^{(1)}z^{-3}X(z) + b_2^{(1)}z^{-2}Y(z) \\ + b_3^{(1)}z^{-3}Y(z) \end{aligned} \tag{3.30}$$

$$\begin{aligned} y(n) = a_0^{(1)}x(n) + a_1^{(1)}x(n-1) + a_2^{(1)}x(n-2) + a_3^{(1)}x(n-3) + b_2^{(1)}y(n-2) \\ + b_3^{(1)}y(n-3). \end{aligned} \tag{3.31}$$

This realization is shown in Figure 4 on page 14.

From Tables 1, 2 and 3 [Ref. 1: pp. 278-279], we have

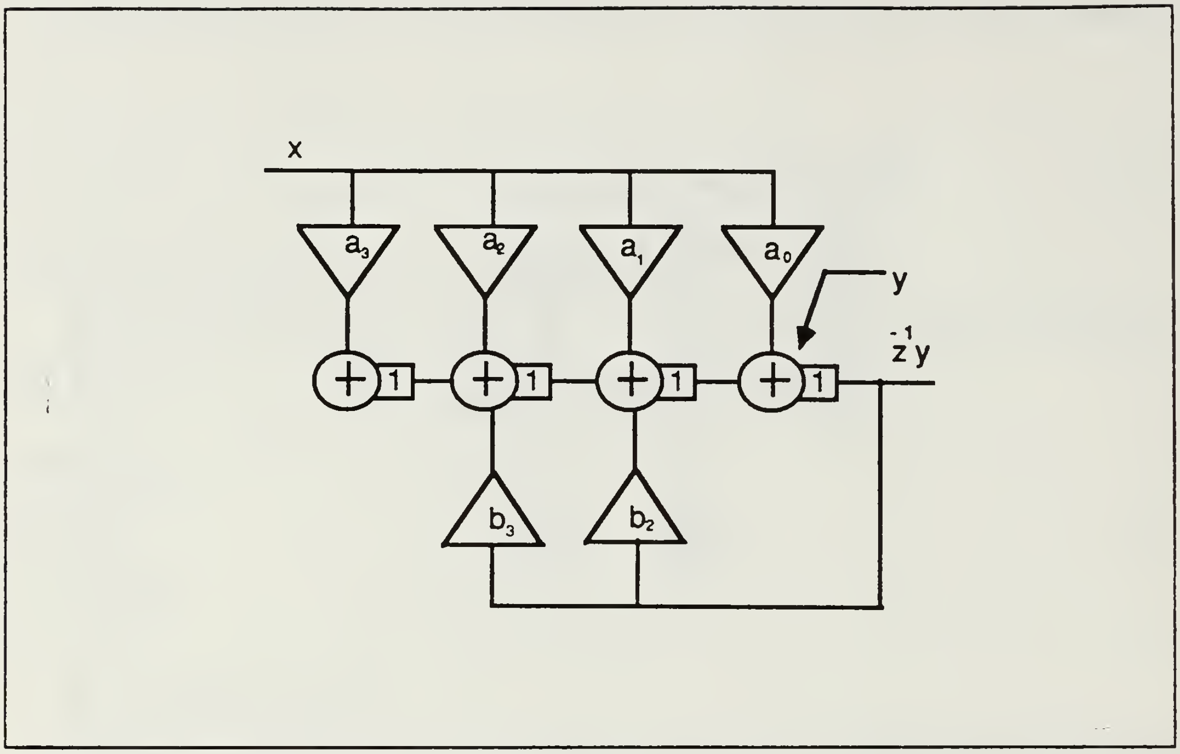

Figure 4. The Augmented IIR Notch Filter: p = 1

$$
\begin{aligned}
&\alpha_0^{(1)} = 1 \\
&\alpha_1^{(1)} = b_1 = 0.32993 \\
&a_0^{(1)} = a_0 = 0.99 \\
&a_1^{(1)} = a_0 b_1 + a_1 = -0.0032993 \\
&a_2^{(1)} = a_1 b_1 + a_2 = 0.8811462 \\
&a_3^{(1)} = a_2 b_1 = 0.3266307 \\
&b_2^{(1)} = b_1^2 + b_2 = -0.8712462 \\
&b_3^{(1)} = b_1 b_2 = -0.32336439\,.
\end{aligned}
\tag{3.32}
$$

The transfer function is

$$
H'(z^{-1}) = \frac{a_0^{(1)} + a_1^{(1)} z^{-1} + a_2^{(1)} z^{-2} + a_3^{(1)} z^{-3}}{1 - b_2^{(1)} z^{-2} - b_3^{(1)} z^{-3}}\,. \tag{3.33}
$$

The difference equation is

$$
\begin{aligned}
y(n) = a_0^{(1)}x(n) + a_1^{(1)}x(n-1) + a_2^{(1)}x(n-2) + a_3^{(1)}x(n-3) + b_2^{(1)}y(n-2) \\
+ b_3^{(1)}y(n-3).
\end{aligned}
\tag{3.34}
$$

D. AUGMENTED IIR NOTCH FILTER: P = 2

From equations (3.12) and (3.13) in section A, we let p = 2. Those equations become:

$$
H'(z^{-1}) = \frac{(a_0 + a_1 z^{-1} + a_2 z^{-2})(1 + \alpha_1^{(2)} z^{-1} + \alpha_2^{(2)} z^{-2})}{(1 - b_1 z^{-1} - b_2 z^{-2})(1 + \alpha_1^{(2)} z^{-1} + \alpha_2^{(2)} z^{-2})}
\tag{3.35}
$$

$$
H'(z^{-1}) = \frac{a_0^{(2)} + a_1^{(2)} z^{-1} + a_2^{(2)} z^{-2} + a_3^{(2)} z^{-3} + a_4^{(2)} z^{-4}}{1 - b_3^{(2)} z^{-3} - b_4^{(2)} z^{-4}}
\tag{3.36}
$$

$$
H'(z^{-1}) = \frac{Y(z)}{X(z)}
\tag{3.37}
$$

$$
\frac{Y(z)}{X(z)} = \frac{a_0^{(2)} + a_1^{(2)} z^{-1} + {}_2^{(2)} z^{-2} + a_3^{(2)} z^{-3} + a_4^{(2)} z^{-4}}{1 - b_3^{(2)} z^{-3} - b_4^{(2)} z^{-4}}
\tag{3.38}
$$

Manipulating (3.38) will result in the difference equation, (3.42).

$$
Y(z)[1 - b_3^{(2)} z^{-3} - b_4^{(2)} z^{-4}] = X(z)[a_0^{(2)} + a_1^{(2)} z^{-1} + a_2^{(2)} z^{-2} + a_3^{(2)} z^{-3} + a_4^{(2)} z^{-4}]
\tag{3.39}
$$

$$
\begin{aligned}
Y(z) - b_3^{(2)} z^{-3} Y(z) - b_4^{(2)} z^{-4} Y(z) = a_0^{(2)} X(z) + a_1^{(2)} z^{-1} X(z) + a_2^{(2)} z^{-2} X(z) \\
+ a_3^{(2)} z^{-3} X(z) + a_4^{(2)} z^{-4} X(z)
\end{aligned}
\tag{3.40}
$$

$$
\begin{aligned}
Y(z) = a_0^{(2)} X(z) + a_1^{(2)} z^{-1} X(z) + a_2^{(2)} z^{-2} X(z) + a_3^{(2)} z^{-3} X(z) + a_4^{(2)} z^{-4} X(z) \\
+ b_3^{(2)} z^{-3} Y(z) + b_4^{(2)} z^{-4} Y(z)
\end{aligned}
\tag{3.41}
$$

$$
\begin{aligned}
y(n) = a_0^{(2)} x(n) + a_1^{(2)} x(n-1) + a_2^{(2)} x(n-2) + a_3^{(2)} x(n-3) + a_4^{(2)} x(n-4) \\
+ b_3^{(2)} y(n-3) + b_4^{(2)} y(n-4).
\end{aligned}
\tag{3.42}
$$

This realization is shown in Figure 5 on page 16.

From Tables 1, 2 and 3 [Ref. 1: pp.278-279], we have

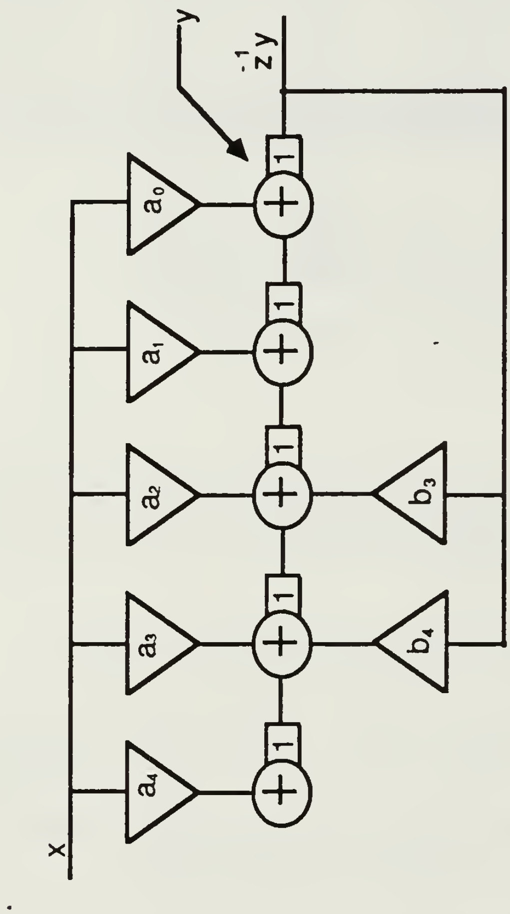

Figure 5. The Augmented IIR Notch Filter: $p = 2$

$$
\begin{aligned}
\alpha_0^{(2)} &= 1 \\
\alpha_1^{(2)} &= b_1 = 0.32993 \\
\alpha_2^{(2)} &= b_1^2 + b_2 = -0.8712462 \\
a_0^{(2)} &= a_0 = 0.99 \\
a_1^{(2)} &= a_0 b_1 + a_1 = -0.0032993 \\
a_2^{(2)} &= a_0(b_1^2 + b_2) + a_1 b_1 + a_2 = 0.01861246 \\
a_3^{(2)} &= a_1(b_1^2 + b_2) + a_2 b_1 = 0.61408096 \\
a_4^{(2)} &= a_2(b_1^2 + b_2) = -0.86253373 \\
b_3^{(2)} &= b_1^3 + 2b_1 b_2 = -0.61081465 \\
b_4^{(2)} &= b_1^2 b_2 + b_2^2 = 0.8539084\,.
\end{aligned}
\tag{3.43}
$$

The transfer function is

$$
H'(z^{-1}) = \frac{a_0^{(2)} + a_1^{(2)} z^{-1} + a_2^{(2)} z^{-2} + a_3^{(2)} z^{-3} + a_4^{(2)} z^{-4}}{1 - b_3^{(2)} z^{-3} - b_4^{(2)} z^{-4}}\,. \tag{3.44}
$$

The difference equation is

$$
\begin{aligned}
y(n) = {} & a_0^{(2)} x(n) + a_1^{(2)} x(n-1) + a_2^{(2)} x(n-2) + a_3^{(2)} x(n-3) \\
& + a_4^{(2)} x(n-4) + b_3^{(2)} y(n-3) + b_4^{(2)} y(n-4).
\end{aligned}
\tag{3.45}
$$

E. AUGMENTED IIR NOTCH FILTER: P = 3

From equations (3.12) and (3.13) in section A, we let p = 3. Those equations become:

$$
H'(z^{-1}) = \frac{(a_0 + a_1 z^{-1} + a_2 z^{-2})(1 + \alpha_1^{(3)} z^{-1} + \alpha_2^{(3)} z^{-2} + \alpha_3^{(3)} z^{-3})}{(1 - b_1 z^{-1} - b_2 z^{-2})(1 + \alpha_1^{(3)} z^{-1} + \alpha_2^{(3)} z^{-2} + \alpha_3^{(3)} z^{-3})} \tag{3.46}
$$

$$
H'(z^{-1}) = \frac{a_0^{(3)} + a_1^{(3)} z^{-1} + a_2^{(3)} z^{-2} + a_3^{(3)} z^{-3} + a_4^{(3)} z^{-4} + a_5^{(3)} z^{-5}}{1 - b_4^{(3)} z^{-4} - b_5^{(3)} z^{-5}} \tag{3.47}
$$

$$
H'(z^{-1}) = \frac{Y(z)}{X(z)} \tag{3.48}
$$

$$
\frac{Y(z)}{X(z)} = \frac{a_0^{(3)} + a_1^{(3)} z^{-1} + a_2^{(3)} z^{-2} + a_3^{(3)} z^{-3} + a_4^{(3)} z^{-4} + a_5^{(3)} z^{-5}}{1 - b_4^{(3)} z^{-4} - b_5^{(3)} z^{-5}} \tag{3.49}
$$

Manipulating (3.49) will result in the difference equation, (3.53).

$$\begin{aligned} &Y(z)[1 - b_4^{(3)}z^{-4} - b_5^{(3)}z^{-5}] = \\ &X(z)[a_0^{(3)} + a_1^{(3)}z^{-1} + a_2^{(3)}z^{-2} + a_3^{(3)}z^{-3} + a_4^{(3)}z^{-4} + a_5^{(3)}z^{-5}] \end{aligned} \tag{3.50}$$

$$\begin{aligned} Y(z) - b_4^{(3)}z^{-4}Y(z) - b_5^{(3)}z^{-5}Y(z) = a_0^{(3)}X(z) &+ a_1^{(3)}z^{-1}X(z) + a_2^{(3)}z^{-2}X(z) \\ &+ a_3^{(3)}z^{-3}X(z) + a_4^{(3)}z^{-4}X(z) \\ &+ a_5^{(3)}z^{-5}X(z) \end{aligned} \tag{3.51}$$

$$\begin{aligned} Y(z) = a_0^{(3)}X(z) &+ a_1^{(3)}z^{-1}X(z) + a_2^{(3)}z^{-2}X(z) + a_3^{(3)}z^{-3}X(z) + a_4^{(3)}z^{-4}X(z) \\ &+ a_5^{(3)}z^{-5}X(z) + b_4^{(3)}z^{-4}Y(z) + b_5^{(3)}z^{-5}Y(z) \end{aligned} \tag{3.52}$$

$$\begin{aligned} y(n) = a_0^{(3)}x(n) &+ a_1^{(3)}x(n-1) + a_2^{(3)}x(n-2) + a_3^{(3)}x(n-3) + a_4^{(3)}x(n-4) \\ &+ a_5^{(3)}x(n-5) + b_4^{(3)}y(n-4) + b_5^{(3)}y(n-5). \end{aligned} \tag{3.53}$$

This realization is shown in Figure 6 on page 19.

From Tables 1, 2 and 3 [Ref. 1: pp. 278-279], we have

$$\begin{aligned} \alpha_0^{(3)} &= 1 \\ \alpha_1^{(3)} &= b_1 = 0.32993 \\ \alpha_2^{(3)} &= b_1^2 + b_2 = -0.8712462 \\ \alpha_3^{(3)} &= b_1^3 + 2b_1b_2 = -0.61081465 \\ a_0^{(3)} &= a_0 = 0.99 \\ a_1^{(3)} &= a_0b_1 + a_1 = -0.0032993 \\ a_2^{(3)} &= a_0(b_1^2 + b_2) + a_1b_1 + a_2 = 0.01861246 \\ a_3^{(3)} &= a_0(b_1^3 + 2b_1b_2) + a_1(b_1^2 + b_2) + a_2b_1 = 0.00937445 \\ a_4^{(3)} &= a_1(b_1^3 + 2b_1b_2) + a_2(b_1^2 + b_2) = -0.66100766 \\ a_5^{(3)} &= a_2(b_1^3 + 2b_1b_2) = -0.6047065 \\ b_4^{(3)} &= b_1^4 + 3b_1^2b_2 + b_2^2 = 0.65238232 \\ b_5^{(3)} &= b_1^3b_2 + 2b_1b_2^2 = 0.59865944\,. \end{aligned} \tag{3.54}$$

The transfer function is

$$H'(z^{-1}) = \frac{a_0^{(3)} + a_1^{(3)}z^{-1} + a_2^{(3)}z^{-2} + a_3^{(3)}z^{-3} + a_4^{(3)}z^{-4} + a_5^{(3)}z^{-5}}{1 - b_4^{(3)}z^{-4} - b_5^{(3)}z^{-5}}\,. \tag{3.55}$$

The difference equation is

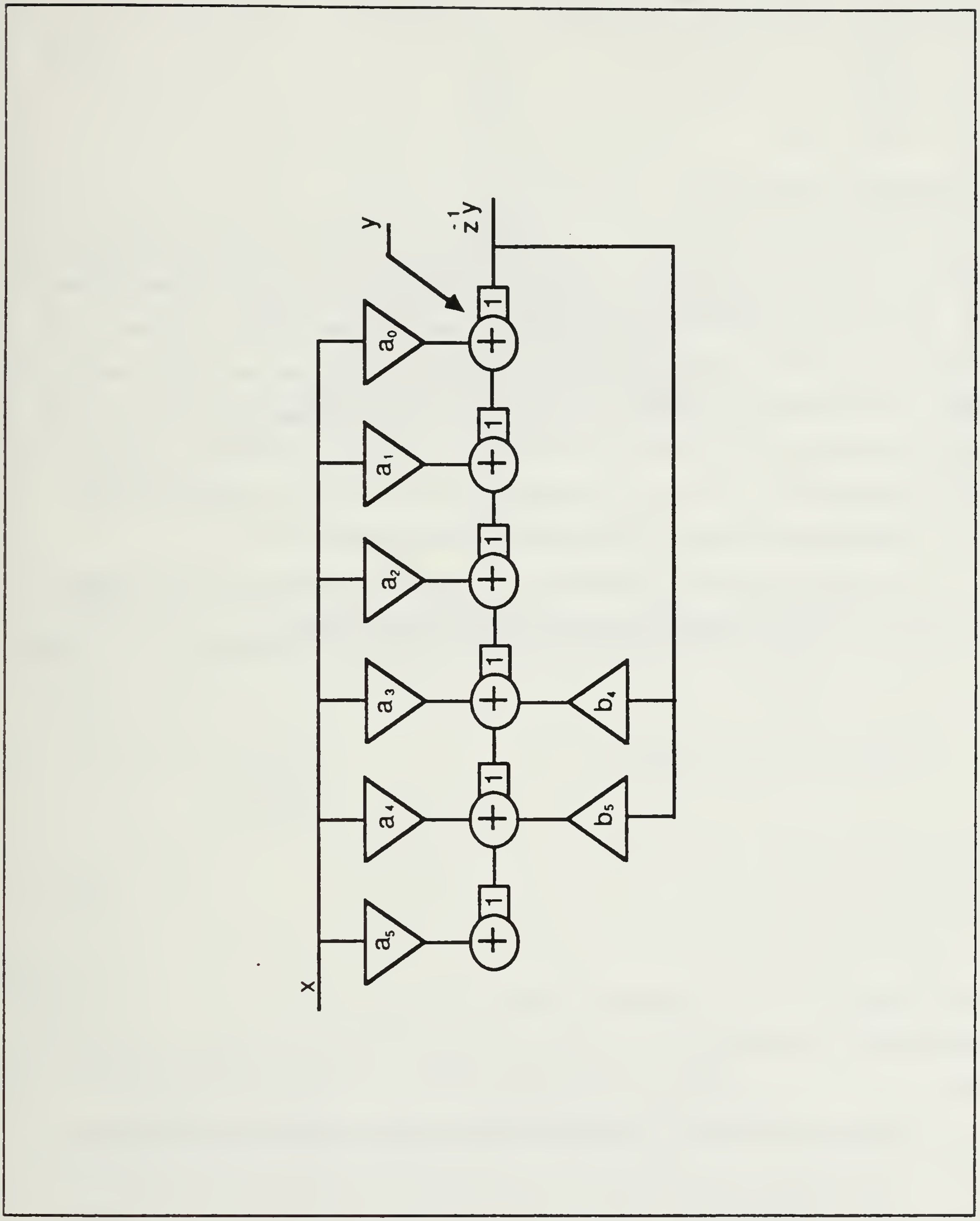

Figure 6. The Augmented IIR Notch Filter: p = 3

$$
\begin{aligned}
y(n) = {} & a_0^{(3)}x(n) + a_1^{(3)}x(n-1) + a_2^{(3)}x(n-2) \\
& + a_3^{(3)}x(n-3) + a_4^{(3)}x(n-4) \\
& + a_5^{(3)}x(n-5) + b_4^{(3)}y(n-4) \\
& + b_5^{(3)}y(n-5).
\end{aligned} \tag{3.56}
$$

F. STABILITY TESTING

From section A if we start with a stable filter, we would like to be assured that the poles of the augmented-order filter all be outside the unit circle in the z^{-1} plane. We desired this because, even though the added poles are cancelled by the zeroes of $(1 + \alpha^{(p)}z^{-1} + \cdots + \alpha^{(p)}z^{-p})$, realization imperfections will prevent exact cancellation and the augmented filter would be unstable. In order to examine the stability question, we will use Jury's stability test as applied to successively higher-degree denominator polynomials in z as p increases [Ref. 1: p. 281]. The results from this paper are summarized below:

1. Condition for Stability of the Original IIR Notch Filter: p = 0

The original IIR filter is described by equation (3.14) and to test for stability, we analyze the following characteristic polynomial resulting from it:

$$F(z) = z^2 - b_1 z - b_2 = 0 \tag{3.57}$$

The conditions for stability are:

$$
\begin{aligned}
1 - b_1 - b_2 > 0 \ldots \text{①} \\
1 + b_1 - b_2 > 0 \ldots \text{②} \\
|b_2| < 1 \ldots \text{③}
\end{aligned} \tag{3.58}
$$

These conditions are shown graphically in Figure 7 on page 21.

Circled numbers in the figure refer to the conditions of the same number in equation (3.58)

Checking for stability of our example IIR notch filter; we have from ① of (3.58)

$$
\begin{aligned}
1 - b_1 - b_2 &> 0 \\
1 - (0.32993) - (-0.9801) &> 0 \\
1.65017 &> 0
\end{aligned}
$$

from ② of (3.58)

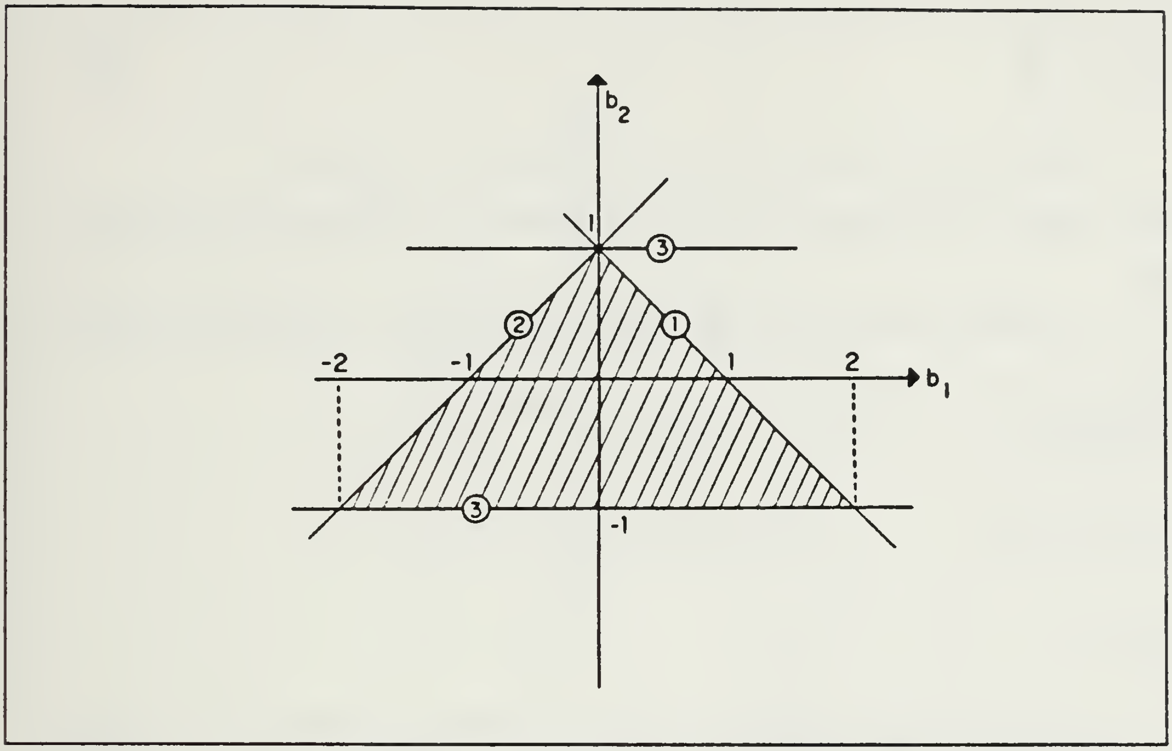

Figure 7. Condition for Stability of the Original Filter [Ref. 1: p. 283]

$$1 + b_1 - b_2 > 0$$
$$1 + (0.32993) - (-0.9801) > 0$$
$$2.31003 > 0$$

from ③ of (3.58)

$$|b_2| < 1$$
$$|0.9801| < 1$$
$$0.9801 < 1$$

Thus the conditions are all satisfied and the original filter is stable.

2. Condition for Stability of the Augmented IIR Notch Filter: p = 1

The augmented IIR notch filter, p = 1, is described by equation (3.33) and to test for stability, we analyze the following characteristic polynomial resulting from it:

$$F(z) = z + b_1 = 0 \tag{3.59}$$

The conditions for stability are:

$$1 + b_1 > 0 \ldots ①$$
$$1 - b_1 > 0 \ldots ② \qquad (3.60)$$

These conditions are shown graphically in Figure 8 on page 23. Circled numbers in the figure refer to the conditions of the same number in equation (3.60)

Checking for stability of our example IIR notch filter; we have from ① of (3.60)

$$\begin{aligned} 1 + b_1 &> 0 \\ 1 + 0.32993 &> 0 \\ 1.32993 &> 0 \end{aligned}$$

from ② of (3.60)

$$\begin{aligned} 1 - b_1 &> 0 \\ 0.67007 &> 0 \end{aligned}$$

Thus the conditions are all satisfied and the augmented IIR notch filter, p = 1, is stable.

3. Condition for Stability of the Augmented IIR Notch Filter: p = 2

The augmented IIR notch filter, p=2, is described by equation (3.44) and to test for stability, we analyze the following characteristic polynomial resulting from it:

$$F(z) = z^2 + b_1 z + (b_1^2 + b_2) \qquad (3.61)$$

The conditions for stability are:

$$\begin{aligned} 1 + b_1 + (b_1^2 + b_2) &> 0 \ldots ① \\ 1 - b_1 + (b_1^2 + b_2) &> 0 \ldots ② \qquad (3.62) \\ |b_1^2 + b_2| &< 1 \ldots ③ \end{aligned}$$

These conditions are shown graphically in Figure 9 on page 24. Circled numbers in the figure refer to the conditions of the same number in equation (3.62)

Checking for stability of our example IIR notch filter; we have from ① of (3.62)

$$\begin{aligned} 1 + b_1 + (b_1^2 + b_2) &> 0 \\ 1 + (0.32993) + [(0.32993)^2 + (-0.9801)] &> 0 \\ 0.4586838 &> 0 \end{aligned}$$

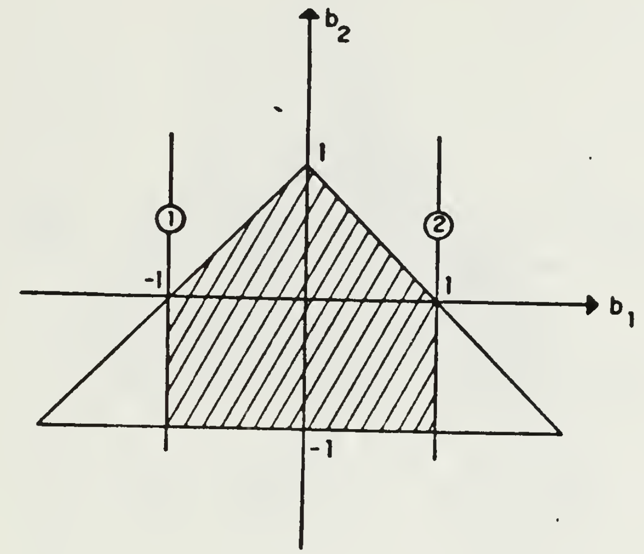

Figure 8. Condition for Stability of the Augmented Filter: p = 1 [Ref. 1: p. 284]

from ② of (3.62)

$$1 - b_1 + (b_1^2 + b_2) > 0$$
$$1 - (0.32993) + [(0.32993)^2 + (-0.9801)] < 0$$
$$-0.2011762 < 0$$

therefore the filter is unstable, from ③ of (3.62)

$$|b_1^2 + b_2| < 1$$
$$|(0.32993)^2 + (-0.9801)| < 1$$
$$0.8712462 < 1$$

Note that ② is violated and here the realization will be unstable.

4. Condition for Stability of the Augmented IIR Notch Filter: p = 3

The augmented IIR notch filter, p = 3, is described by equation (3.55) and to test for stability, we analyze the following characteristic polynomial resulting from it:

$$F(z) = z^3 + b_1 z^2 + (b_1^2 + b_2)z + (b_1^3 + 2b_1 b_2) \tag{3.63}$$

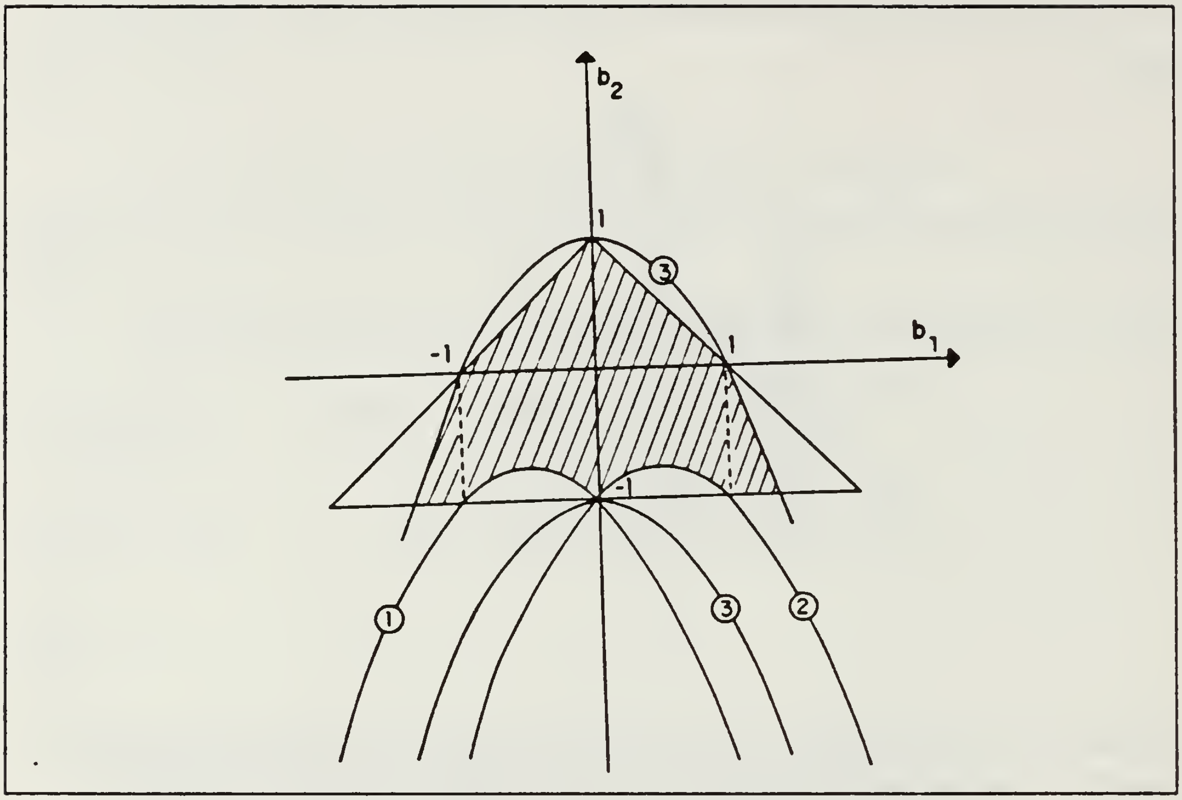

Figure 9. Condition for Stability of the Augmented Filter: p = 2 [Ref. 1: p. 285]

The conditions for stability are:

$$
\begin{aligned}
1 + b_1 + (b_1^2 + b_2) + (b_1^3 + 2b_1b_2) > 0 \ldots \text{①} \\
1 - b_1 + (b_1^2 + b_2) - (b_1^3 + 2b_1b_2) > 0 \ldots \text{②} \\
|b_1^3 + 2b_1b_2| < 1 \ldots \text{③} \\
|(b_1^3 + 2b_1b_2)^2 - 1| > |b_1(b_1^3 + 2b_1b_2) - (b_1^2 + b_2)| \ldots \text{④}
\end{aligned}
\qquad (3.64)
$$

These conditions are shown graphically in Figure 10 on page 25. Circled numbers in the figure refer to the conditions of the same number in equation (3.64)

Checking for stability of our example IIR notch filter; we have from ① of (3.64)

$$
\begin{aligned}
1 + b_1 + (b_1^2 + b_2) + (b_1^3 + 2b_1b_2) > 0 \\
1 + 0.32993 + [(0.32993)^2 + (-0.9801)] + [(0.32993)^3 + 2(0.32993)(-0.9801)] > 0 \\
-0.15213085 < 0
\end{aligned}
$$

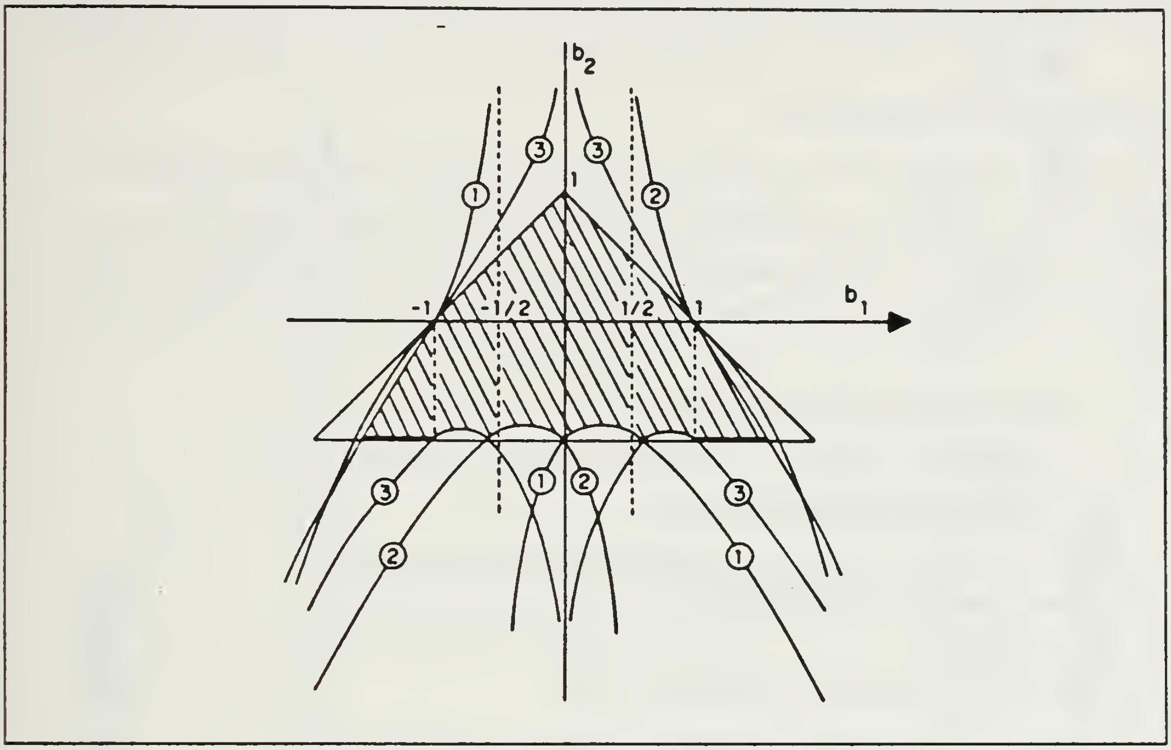

Figure 10. Condition for Stability of the Augmented Filter: **p = 3** [Ref. 1: p. 286]

therefore the filter is unstable, from ② of (3.64)

$$1 - b_1 + (b_1^2 + b_2) - (b_1^3 + 2b_1b_2) > 0$$
$$1 - 0.32993 + [(0.32993)^2 + (-0.9801)] - [(0.32993)^3 + 2(0.32993)(-0.9801)] > 0$$
$$0.40963846 > 0$$

from ③ of (3.64)

$$|b_1^3 + 2b_1b_2| < 1$$
$$|(0.32993)^3 + 2(0.32993)(-0.9801)| < 1$$
$$0.61081465 < 1$$

from ④ of (3.64)

$$|(b_1^3 + 2b_1b_2)^2 - 1| > |b_1(b_1^3 + 2b_1b_2) - (b_1^2 + b_2)|$$
$$|\{(0.32993)^3 + 2(0.32993)(-0.9801)\}^2 - 1| <$$
$$|(0.32993)\{(0.32993)^3 + 2(0.32993)(-0.9801)\} - \{(0.32993)^2 + (-0.9801)\}|$$
$$0.62690546 < 0.66972012$$

therefore the filter is unstable.

Note that ① and ④ are violated and therefore the realization will be unstable.

G. FREQUENCY RESPONSE

From the transfer functions (3.22), (3.33), (3.44), and (3.55), we use the program in Appendix A to calculate the magnitude and phase of the frequency response of the several realizations. We plot the magnitude of these responses for the original filter and for the pipeline augmented that all of the frequency responses are the same from those figures.

1. Second Order IIR Notch Filter: p = 0

The magnitude response is shown in Figure 11 on page 27.

2. The Augmented IIR Notch Filter: p = 1

The magnitude response is shown in Figure 12 on page 28.

3. The Augmented IIR Notch Filter: p = 2

The magnitude response is shown in Figure 13 on page 29.

4. The Augmented IIR Notch Filter: p = 3

The magnitude response is shown in Figure 14 on page 30.

We know from section F that the example augmented IIR notch filter for p = 2, and 3 are unstable but we cannot see that from these magnitude responses. We have to find another way to show which one is stable or unstable. This will be shown in section H.

H. UNIT SAMPLE RESPONSE

From the difference equations (3.23), (3.34), (3.45), and (3.56), we use the program in Appendix B to calculate the output for a unit sample input to each of the pipeline realizations. We plot the outputs in Figures 15, 16, 17 and 18. We can see that the outputs for Figure 15 (p=0) and Figure 16 (p=1) are stable, however the outputs for Figure 17 (p=2) and Figure 18 (p=3) are unstable as predicted by the stability testing in section F.

1. Second Order IIR Notch Filter: p = 0

The output is shown in Figure 15 on page 31.

2. The Augmented IIR Notch Filter: p = 1

The output is shown in Figure 16 on page 32.

3. The Augmented IIR Notch Filter: p = 2

The output is shown in Figure 17 on page 33.

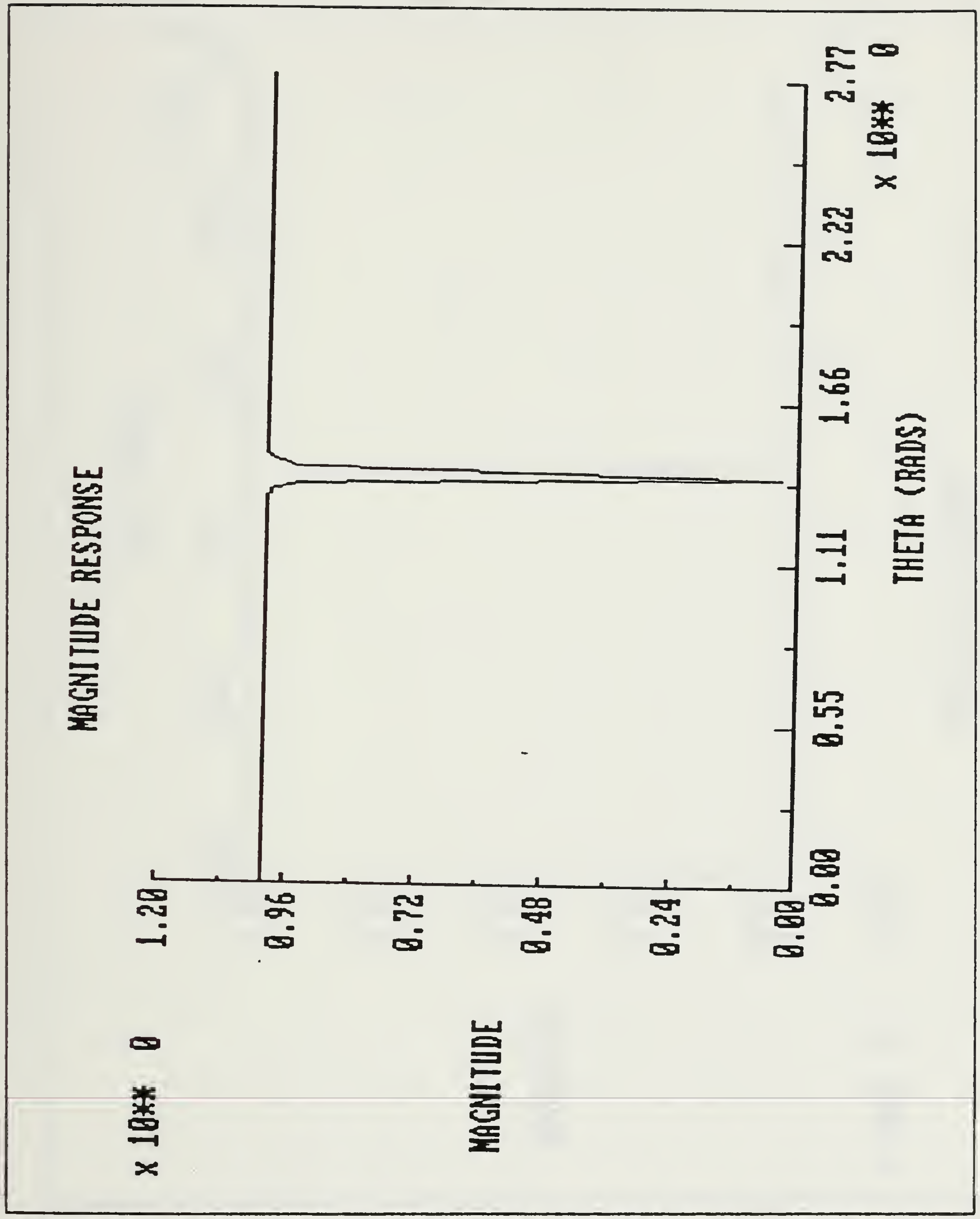

Figure 11. Magnitude Response of Second Order IIR Notch Filter: p = 0

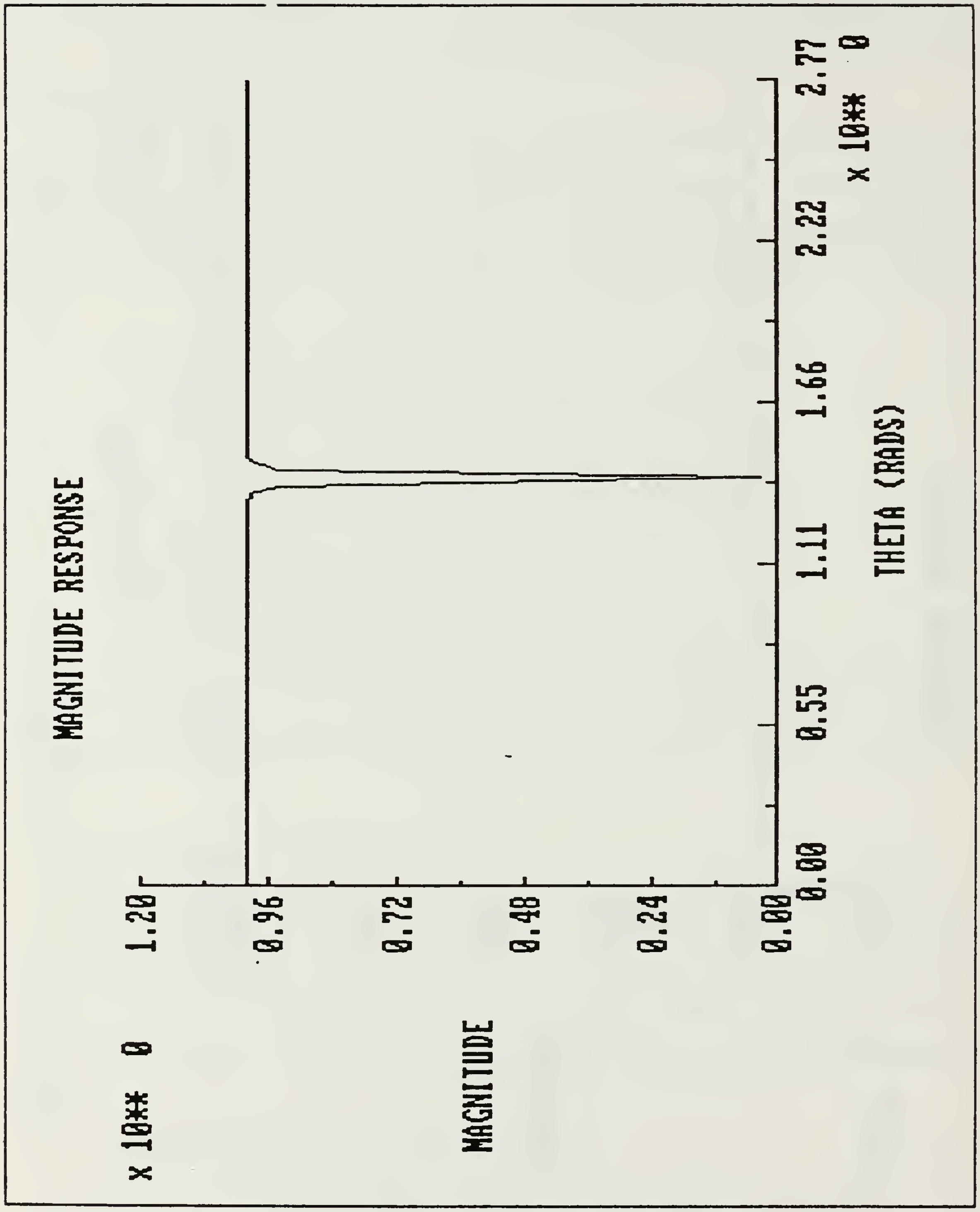

Figure 12. Magnitude Response of the Augmented IIR Notch Filter: p = 1

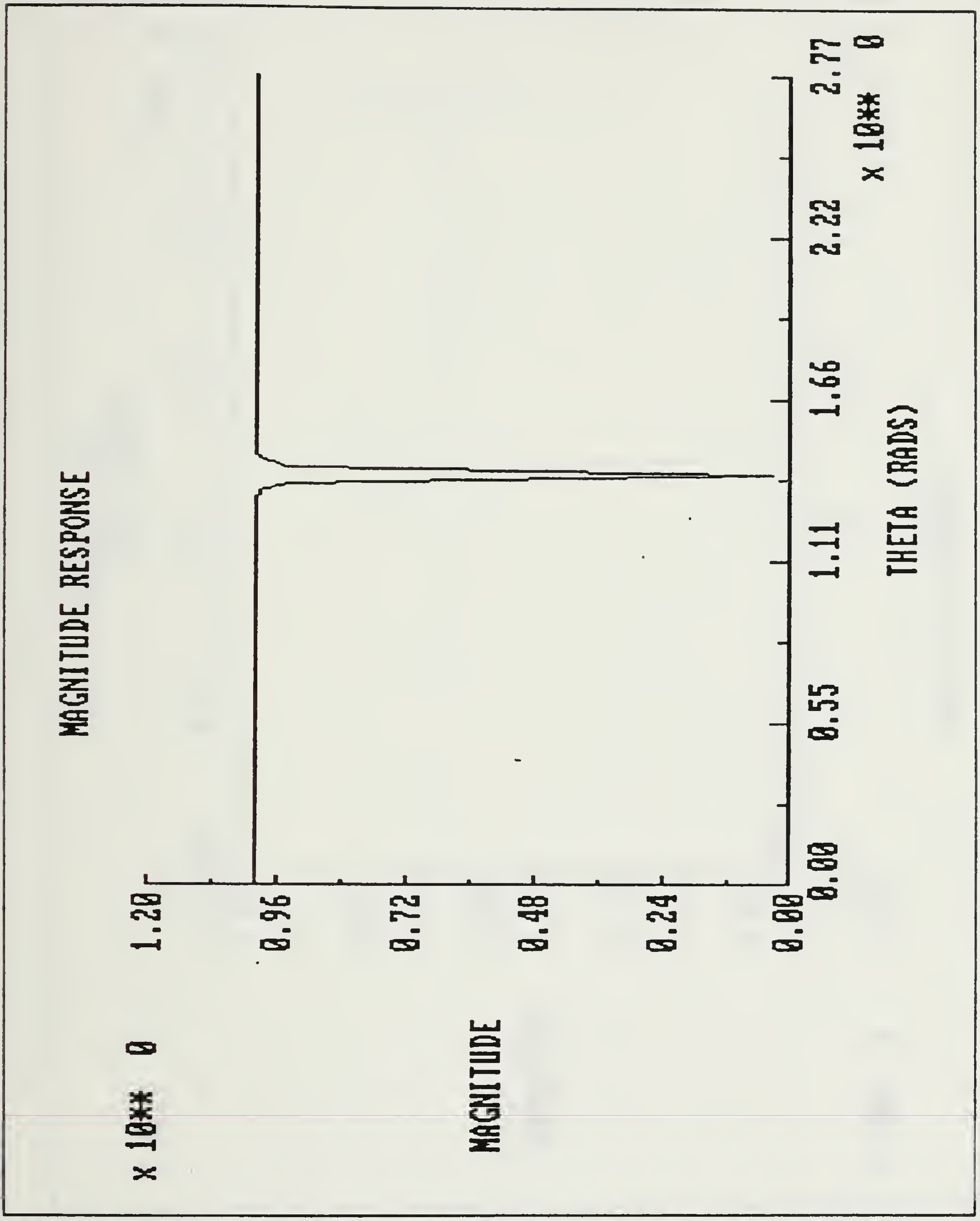

Figure 13. Magnitude Response of the Augmented IIR Notch Filter: p = 2

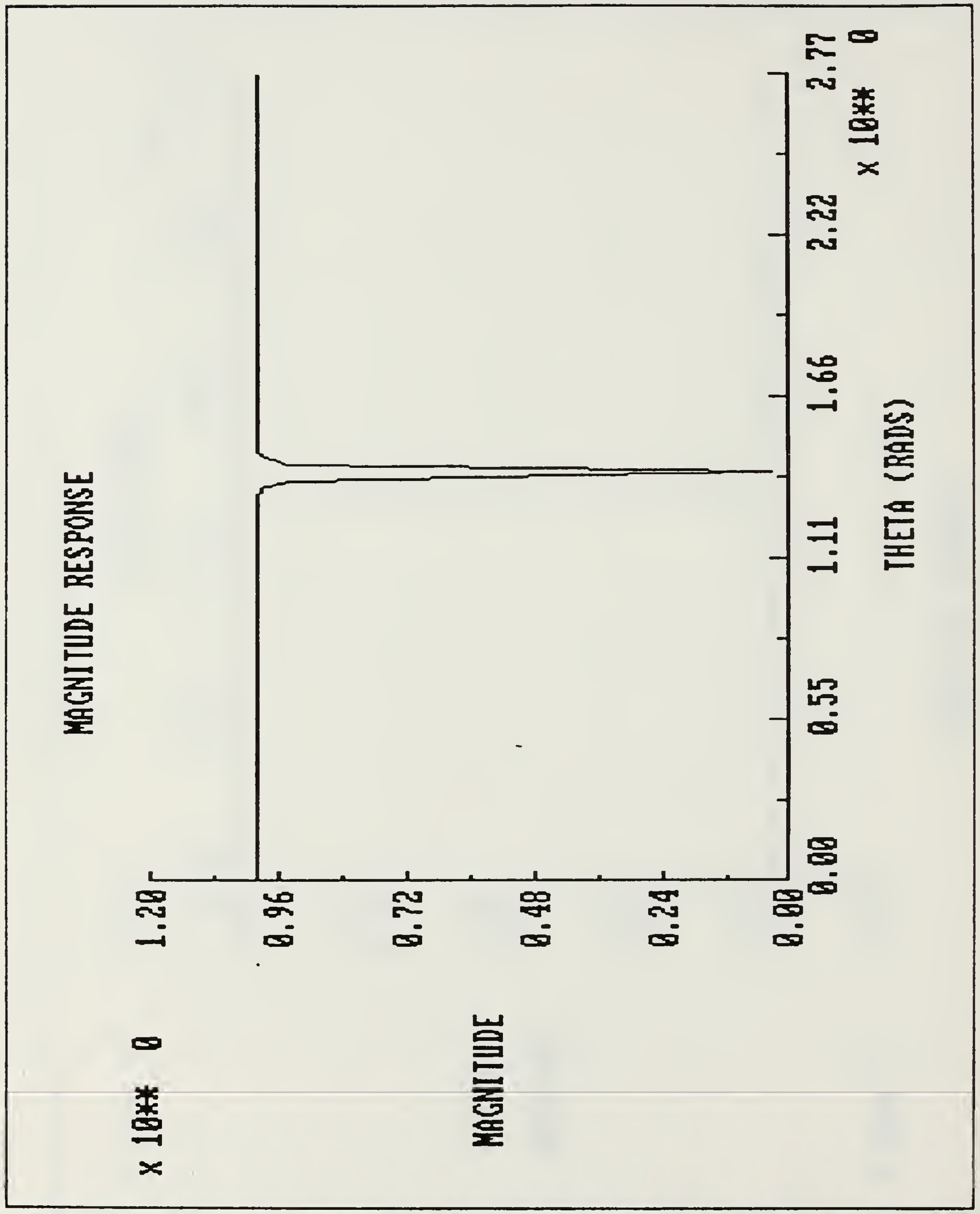

Figure 14. Magnitude Response of the Augmented IIR Notch Filter: p = 3

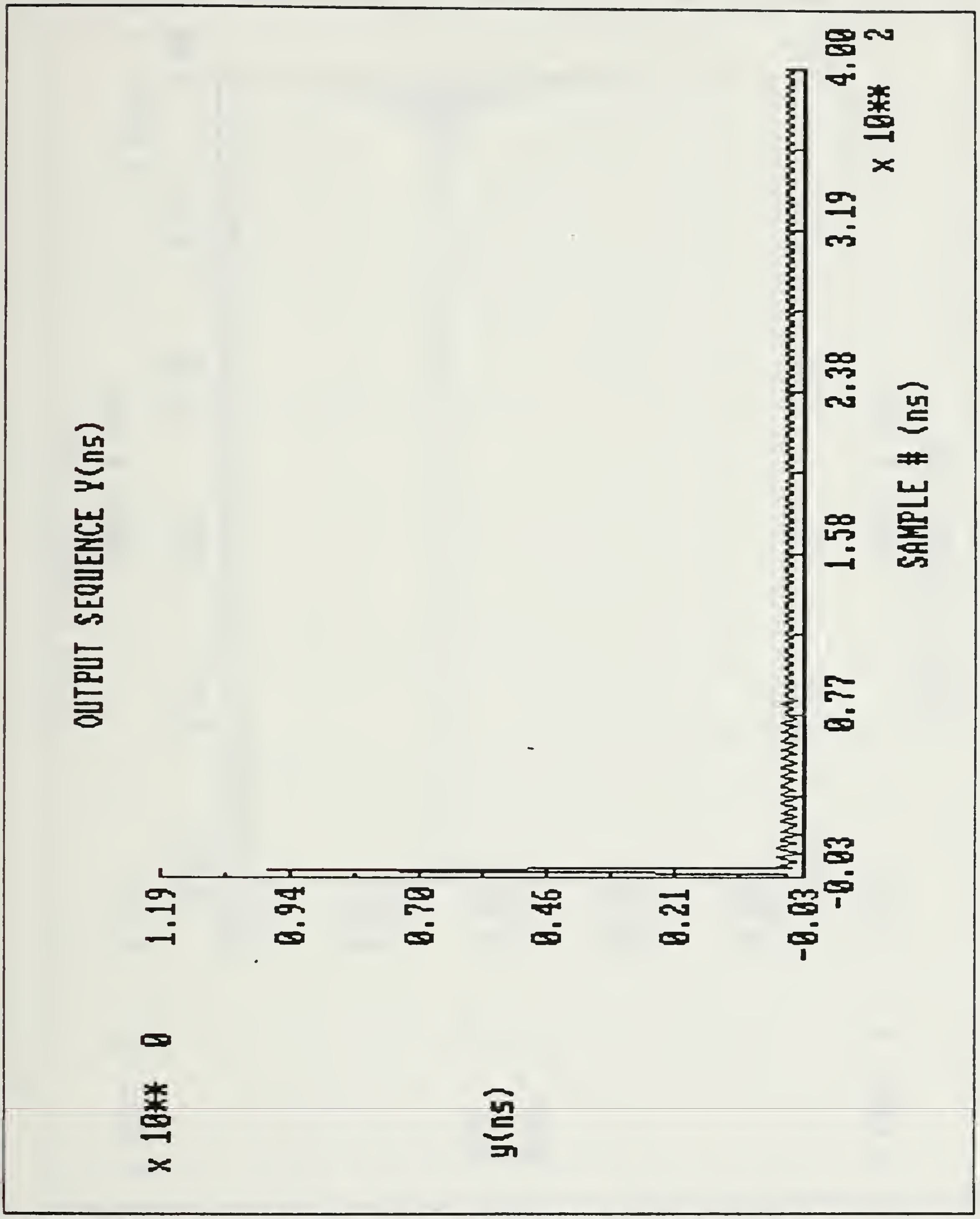

Figure 15. Output Sequence of Second Order IIR Notch Filter: p = 0

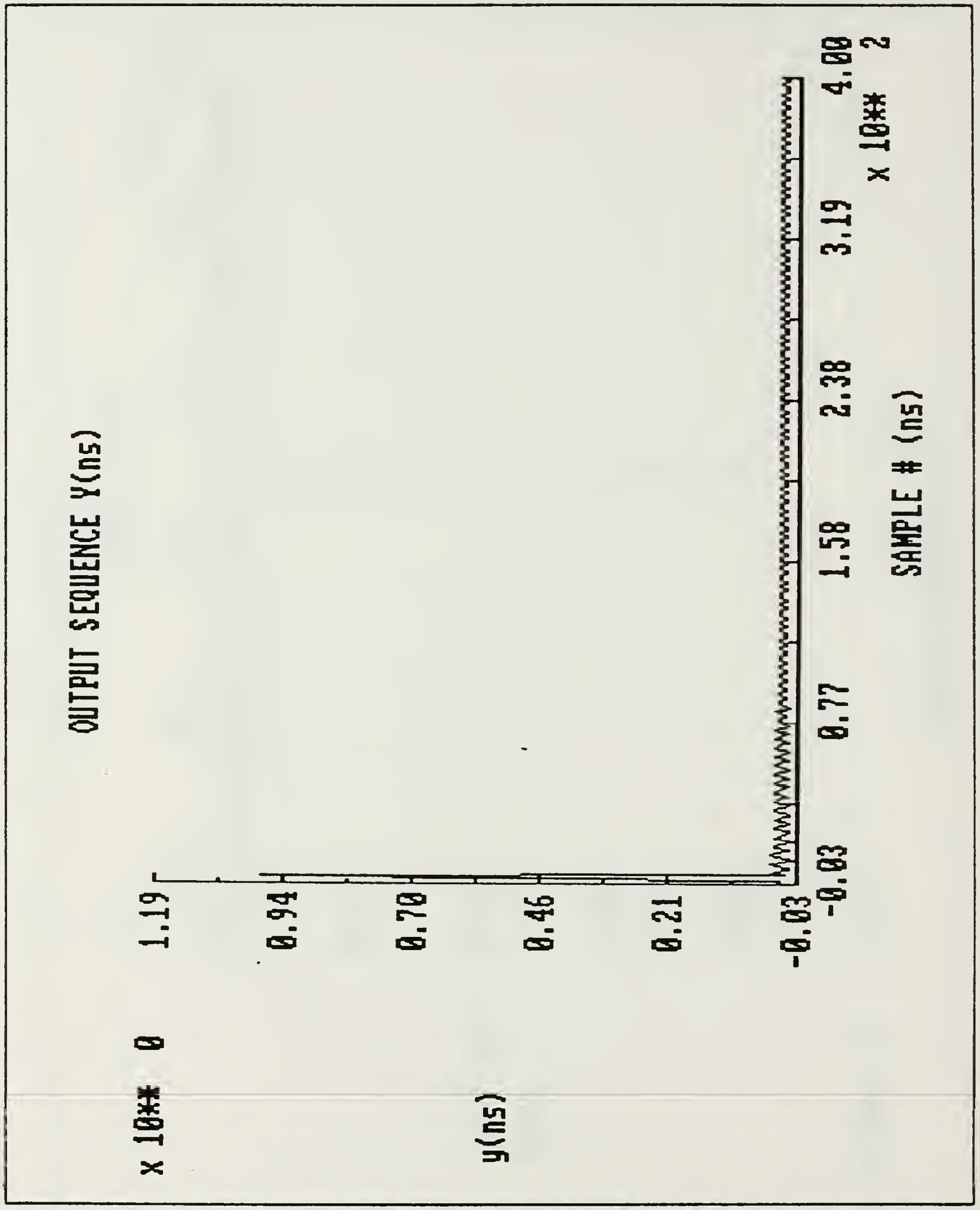

Figure 16. Output Sequence of The Augmented IIR Notch Filter: p = 1

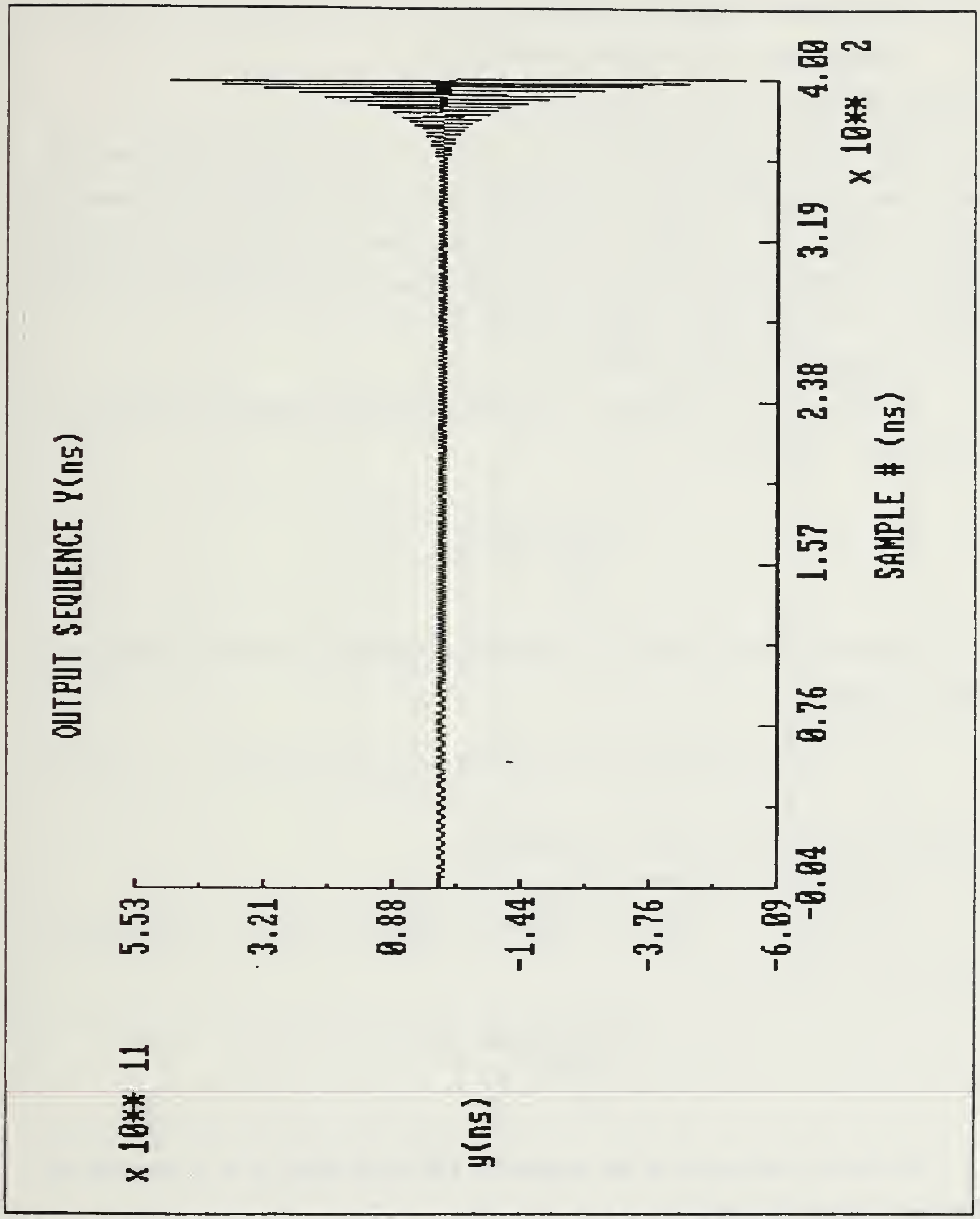

Figure 17. Output Sequence of The Augmented IIR Notch Filter: p = 2

4. The Augmented IIR Notch Filter: p = 3

The output is shown in Figure 18 on page 35.

I. PIPELINE IMPLEMENTATION

All of the IIR filters in sections B, C, D, and E can be developed into high-speed IIR filters by applying pipeline techniques. We use Ka-stage pipeline add units and Km-stage pipeline multipliers to increase the speed of the filters. The Km stages are involved in the coefficient multiplies and the Ka stages are involved in the adds. The increase in order of the filter is $P = 2Ka + Km - 2$ [Ref. 1: p. 280].

1. Second Order IIR Notch Filter: p = 0

To achieve a p=0 realization, we cannot use pipeline modules with longer delays than $Ka = 1$ and $Km = 0$.

$$\begin{aligned} P &= 2Ka + Km - 2 \\ &= 2(1) + 0 - 2 \\ &= 0 \end{aligned}$$

We have a realization of the original IIR notch filter but with an additional 3 delays. The equation is

$$z^{-3}Y = z^{-3}\left[a_0X + a_1z^{-1}X + a_2z^{-2}X + b_1z^{-1}Y + b_2z^{-2}Y\right]. \tag{3.65}$$

This realization is shown in Figure 19 on page 36.

2. The Augmented IIR Notch Filter: p = 1

To achieve a p=1 realization, we cannot use pipeline modules with longer delays than $Ka = 1$ and $Km = 1$.

$$\begin{aligned} P &= 2Ka + Km - 2 \\ &= 2(1) + 1 - 2 \\ &= 1 \end{aligned}$$

We have a realization of the augmented IIR notch filter, p = 1, but with an additional 4 delays. The equation is

$$z^{-4}Y = z^{-4}\left[a_0X + a_1z^{-1}X + a_2z^{-2}X + a_3z^{-3}X + b_2z^{-2}Y + b_3z^{-3}Y\right]. \tag{3.66}$$

This realization is shown in Figure 20 on page 37.

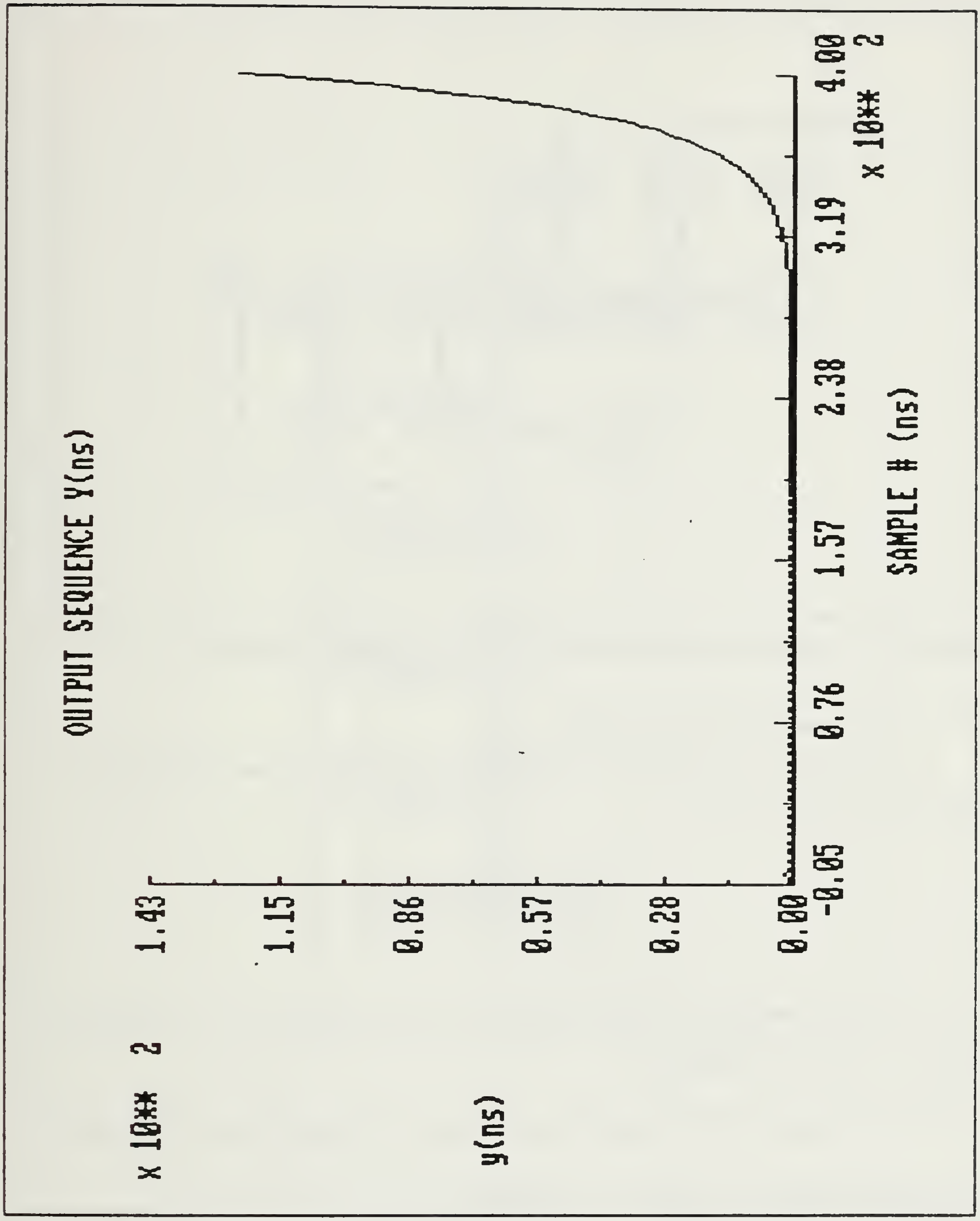

Figure 18. Output Sequence of The Augmented IIR Notch Filter: p = 3

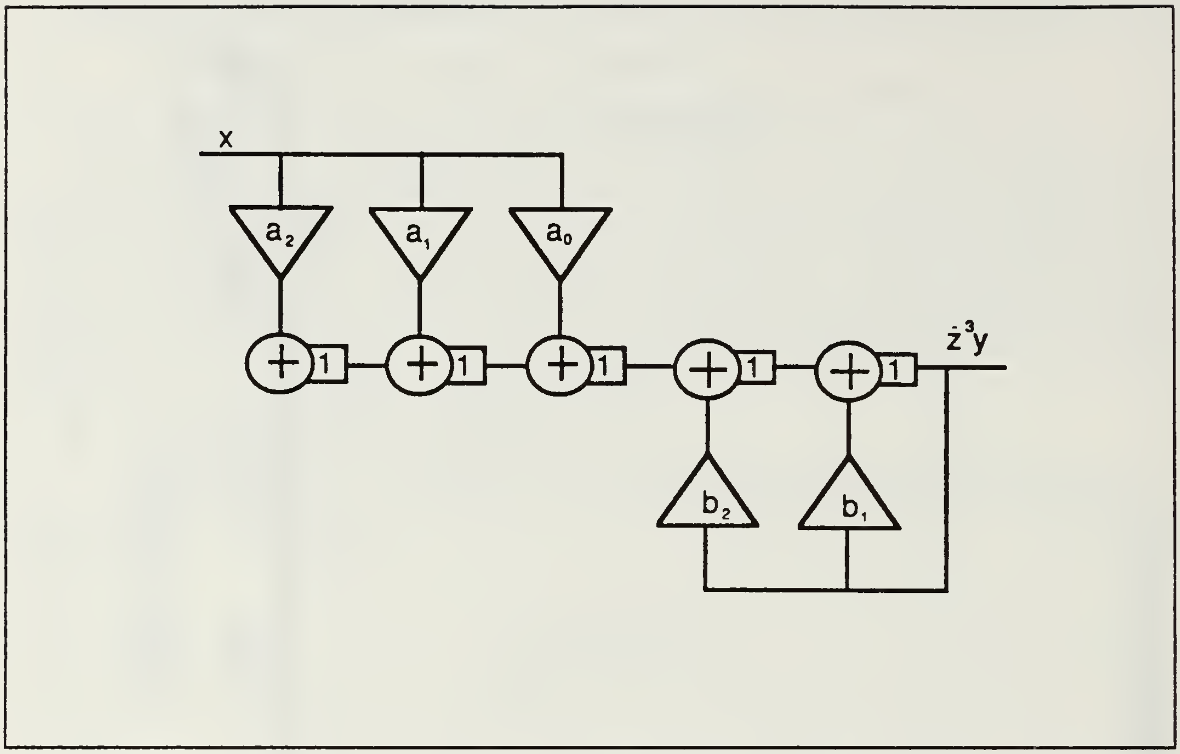

Figure 19. Second Order IIR Notch Filter: p = 0 with Ka = 1, Km = 0

3. The Augmented IIR Notch Filter: p = 2

To achieve a p = 2 realization, we cannot use pipeline modules with longer delays than $Ka = 1$ and $Km = 2$.

$$\begin{aligned} P &= 2Ka + Km - 2 \\ &= 2(1) + 2 - 2 \\ &= 2 \end{aligned}$$

We have a realization of the augmented IIR notch filter, p = 2, but with an additional 5 delays. The equation is

$$z^{-5}Y = z^{-5}\left[a_0X + a_1z^{-1}X + a_2z^{-2}X + a_3z^{-3}X + a_4z^{-4}X + b_3z^{-3}Y + b_4z^{-4}Y\right]. \quad (3.67)$$

This realization is shown in Figure 21 on page 38.

4. The Augmented IIR Notch Filter: p = 3

To achieve a p = 3 realization, we cannot use pipeline modules with longer delays than $Ka = 1$ and $Km = 3$.

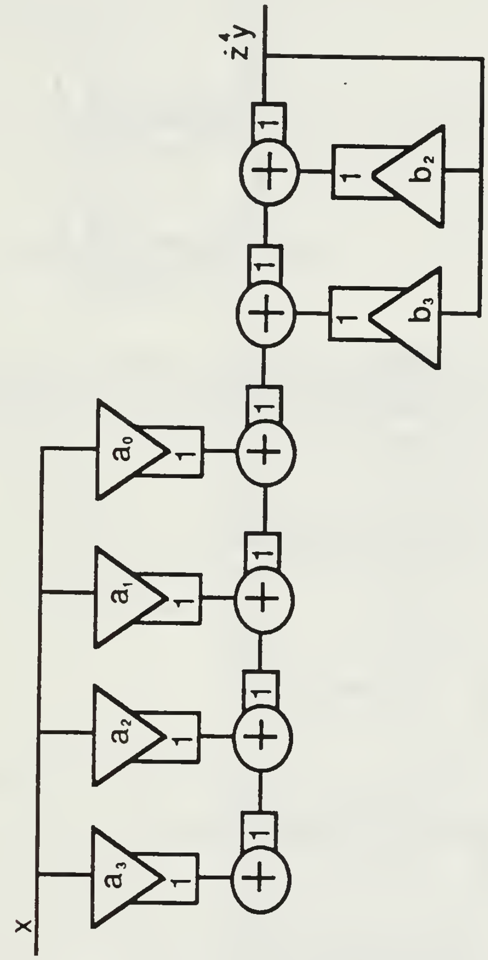

Figure 20. The Augmented IIR Notch Filter: p = 1 with Ka = 1, Km = 1

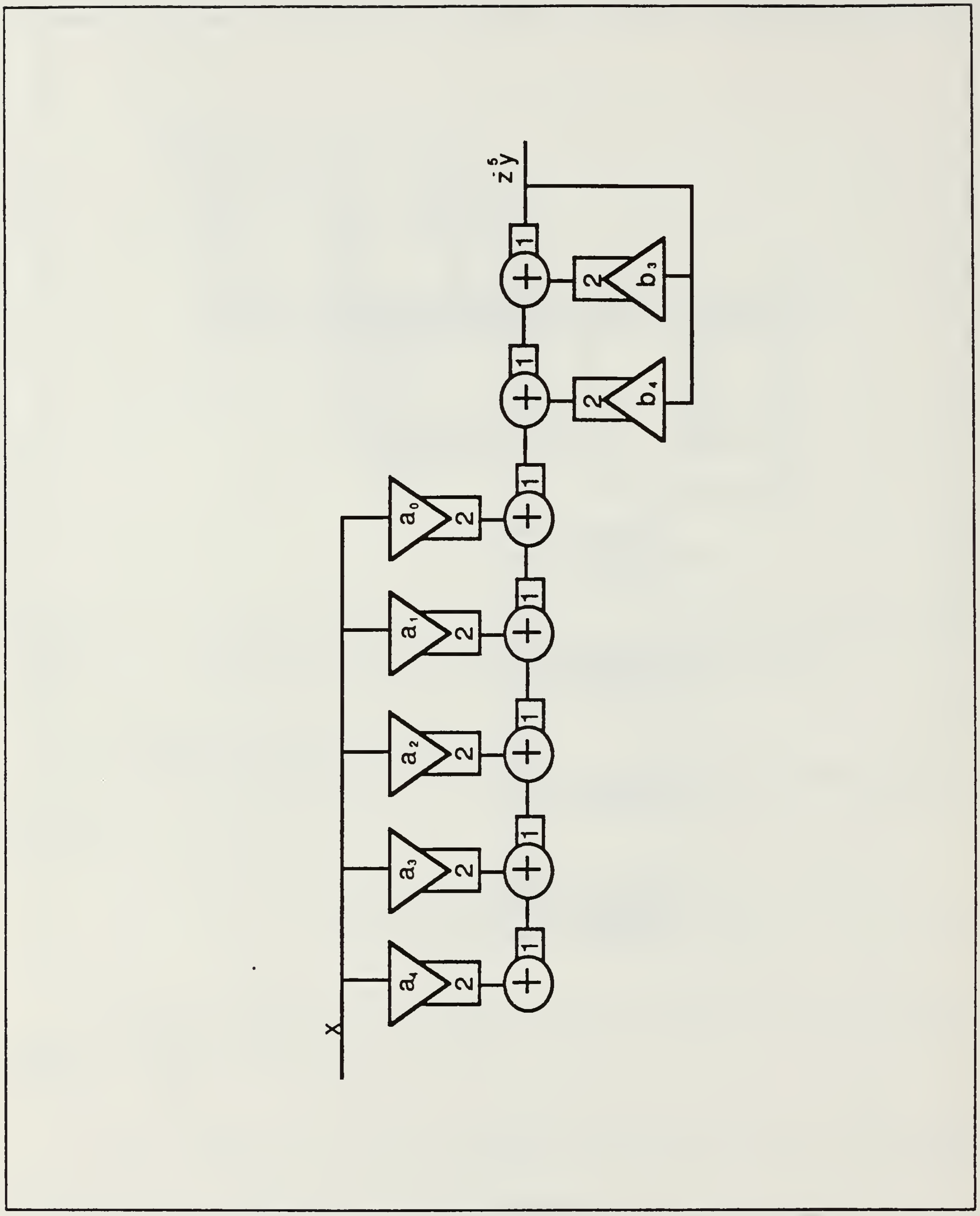

Figure 21. The Augmented IIR Notch Filter: p = 2 with Ka = 1, Km = 2

$$
\begin{aligned}
P &= 2Ka + Km - 2 \\
&= 2(1) + 3 - 2 \\
&= 3
\end{aligned}
$$

We have a realization of the augmented IIR notch filter, p = 3, but with an additional 6 delays. The equation is

$$
z^{-6}Y = z^{-6}\begin{bmatrix} a_0X + a_1z^{-1}X + a_2z^{-2}X + a_3z^{-3}X \\ + a_4z^{-4}X + a_5z^{-5}X + b_4z^{-4}Y + b_5z^{-5}Y \end{bmatrix}. \tag{3.68}
$$

This realization is shown in Figure 22 on page 40.

Table 4. LOOMIS/SINHA PIPELINE RECURSIVE DESIGN

p-order	Ka-add units	Km-multipliers	D (output delayed)	Figure
0	1	0	3	19
1	1	1	4	20
2	1	2	5	21
3	1	3	6	22

J. SUMMARY

This chapter investigated a method for applying pipeline techniques to the design of a high-speed IIR notch filter. Using the Loomis/Sinha method, IIR notch filters operating at rates hitherto impossible can be designed. The general structure of the IIR notch filter and the method of calculating the multiplier coefficients have been presented. The stability of the resulting realization has been investigated and a technique for ascertaining the stability of the realization has been presented. The delay and complexity of each realization are summarized in Table 4.

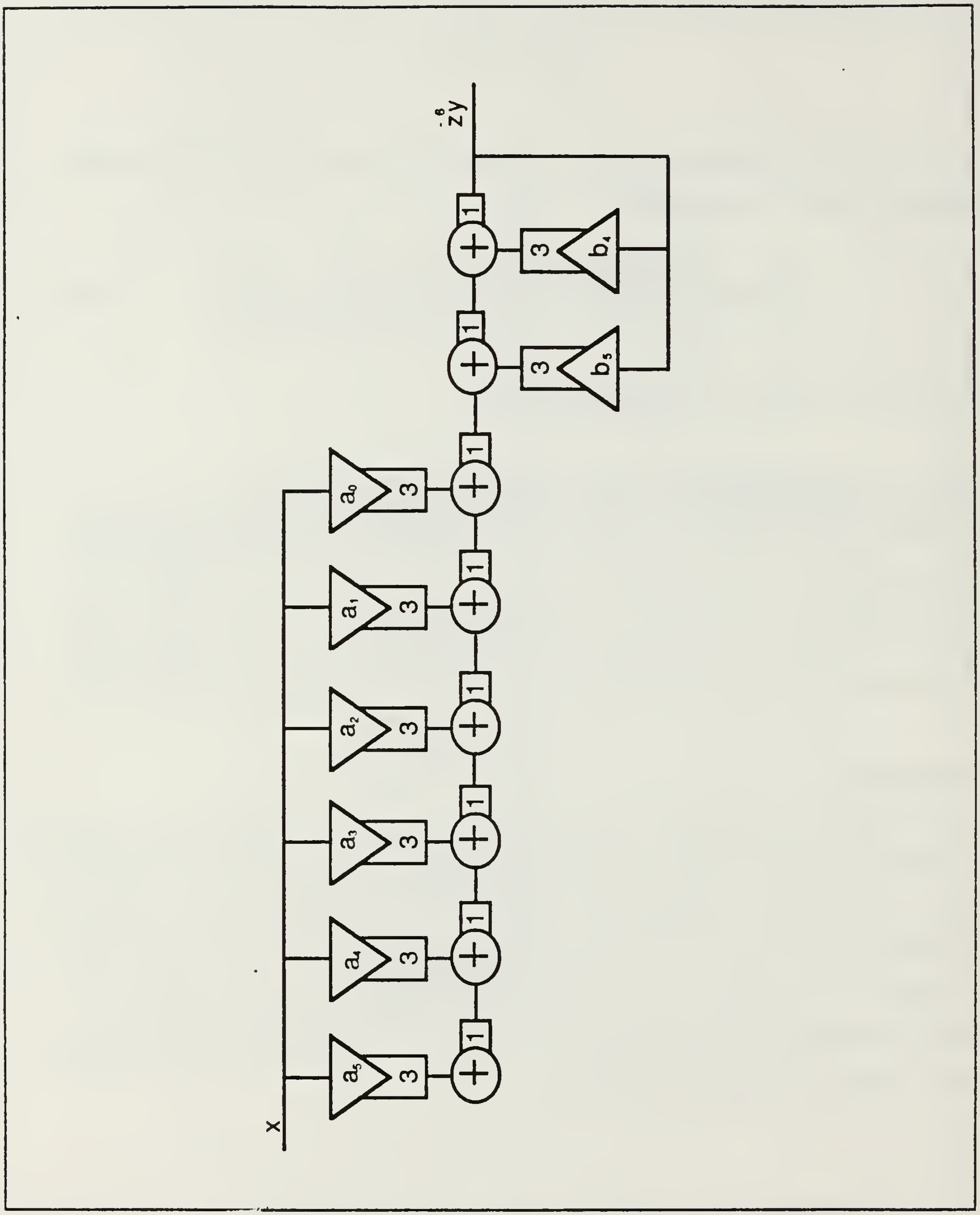

Figure 22. The Augmented IIR Notch Filter: p = 3 with Ka = 1, Km = 3

IV. GNANASEKARAN'S STATE SPACE REALIZATIONS

IIR filter realizations are possible and most of them can be presented within the general time domain representation of the system under consideration. In continuous time it was convenient to represent a linear constant coefficient differential equation by a system of first-order differential equations for analysis purposes. It is, therefore, not surprising that it is useful to represent a linear constant coefficient difference equation by a system of first-order linear constant coefficient difference equations. Such a formulation results in a special form of representation of the difference equation called a state variable representation.

A. STATE SPACE REPRESENTATION OF A NTH ORDER RECURSIVE DIGITAL FILTER

We are used to working with a difference equation of the form

$$y(n) = b_1 y(n-1) + \cdots + b_N y(n-N) + a_0 x(n) + a_1 x(n-1) + \cdots + a_L x(n-M) \quad (4.1)$$

or in the notation

$$y(n) = \sum_{k=1}^{N} b_k y(n-k) + \sum_{k=0}^{M} a_k x(n-k) \quad (4.2)$$

that is called the **Nth-order** form. An alternative is to describe an LTI system with N first-order difference equations and an output equation. This alternative form for describing a linear time-invariant system is known as the **state-space** description and can be written as a matrix equation:

$$\mathbf{v}(n) = \mathbf{A}\mathbf{v}(n-1) + \mathbf{B}x(n) \qquad (\textit{state equation}) \quad (4.3)$$

with the system output given by

$$y(n) = \mathbf{C}^T\mathbf{v}(n) + dx(n) \qquad (\textit{output equation}). \quad (4.4)$$

In Eqs. (4.3) and (4.4), $\mathbf{v}(n)$ is a vector with N rows and 1 column (an $N \times 1$ vector or column matrix) of newly defined variables called the states, $x(n)$ is the system input, and $y(n)$ its output. In the matrix state equation of Eq. (4.3) the matrix **A** must be an

$N \times N$ matrix and the **B** matrix an $N \times 1$ vector. The scalar output $y(n)$ is defined by Eq. (4.4) where **C** is a $N \times 1$ column matrix and d is simply a real number that multiplies (or scales) the input $x(n)$. The elements in **A**, **B**, **C**, and d are constants that depend only upon the system parameters.

B. WRITING THE STATE SPACE EQUATIONS

To illustrate this approach to modeling of LTI systems we return to a version of the Nth-order difference equation described as a second order IIR filter, namely

$$y(n) = b_1 y(n-1) + b_2 y(n-2) + a_0 x(n) + a_1 x(n-1) + a_2 x(n-2) \tag{4.5}$$

$$Y = b_1 z^{-1} Y + b_2 z^{-2} Y + a_0 X + a_1 z^{-1} X + a_2 z^{-2} X. \tag{4.6}$$

We define the state varible as: V_1, V_2. Then, we can establish the following relationships:

$$\begin{aligned} V_1 &= Y = b_1 z^{-1} Y + b_2 z^{-2} Y + a_0 X + a_1 z^{-1} X + a_2 z^{-2} X \\ V_2 &= z^{-1} V_1 = z^{-1} Y \\ z^{-1} V_2 &= z^{-2} Y. \end{aligned} \tag{4.7}$$

Therefore the state equation is:

$$\begin{aligned} V_1 &= b_1 z^{-1} V_1 + b_2 z^{-1} V_2 + a_0 X + a_1 z^{-1} X + a_2 z^{-2} X \\ V_2 &= z^{-1} V_{1.} \end{aligned} \tag{4.8}$$

We express it in the matrix state equation:

$$\mathbf{v}(n) = \mathbf{A}\mathbf{v}(n-1) + \mathbf{B}x(n) \tag{4.9}$$

$$\mathbf{V} = \mathbf{A} z^{-1} \mathbf{V} + \mathbf{B} X \tag{4.10}$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ 1 & 0 \end{bmatrix} z^{-1} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} a_0 + a_1 z^{-1} + a_2 z^{-2} \\ 0 \end{bmatrix} X. \tag{4.11}$$

That means:

$$\mathbf{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{4.12}$$

$$\mathbf{A} = \begin{bmatrix} b_1 & b_2 \\ 1 & 0 \end{bmatrix} \tag{4.13}$$

$$\mathbf{B} = \begin{bmatrix} a_0 + a_1 z^{-1} + a_2 z^{-2} \\ 0 \end{bmatrix}. \tag{4.14}$$

The output equation will be:

$$y(n) = \mathbf{C}^T \mathrm{v}(n) + dx(n) \tag{4.15}$$

$$Y = [c_1 \quad c_2] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + dX \tag{4.16}$$

$$Y = c_1 V_1 + c_2 V_2 + dX. \tag{4.17}$$

C. MODIFIED STATE SPACE EQUATION

From the state equation

$$\mathrm{v}(n) = \mathbf{A}\mathrm{v}(n-1) + \mathbf{B}x(n), \tag{4.18}$$

the computation of v(n) requires the immediate past state vector v(n-1). This recursive nature severely limits the speed that can be achieved by direct implementation of the state equation. The speed limits can be overcome by modifying this state equation so that v(n) depends not on the immediate past vector, $v(n-1)$, but on $v(n-L)$ for some $L > 1$. This can be accomplished as follows:

For $L = 1$ the state equation is

$$\mathrm{v}(n) = \mathbf{A}\mathrm{v}(n-1) + \mathbf{B}x(n) \tag{4.19}$$

$$\mathbf{V} = \mathbf{A}z^{-1}\mathbf{V} + \mathbf{B}X. \tag{4.20}$$

Delaying equation (4.19) by 1, 2, and 3 yields

$$\mathrm{v}(n-1) = \mathbf{A}\mathrm{v}(n-2) + \mathbf{B}x(n-1) \tag{4.21}$$

$$\mathrm{v}(n-2) = \mathbf{A}\mathrm{v}(n-3) + \mathbf{B}x(n-2) \tag{4.22}$$

$$\mathrm{v}(n-3) = \mathbf{A}\mathrm{v}(n-4) + \mathbf{B}x(n-3). \tag{4.23}$$

For $L = 2$ putting (4.21) into (4.22), the state equation, results is

$$v(n) = A[Av(n-2) + Bx(n-1)] + Bx(n) \tag{4.24}$$

$$v(n) = A^2v(n-2) + ABx(n-1) + Bx(n) \tag{4.25}$$

$$V = A^2z^{-2}V + ABz^{-1}X + BX. \tag{4.26}$$

For $L = 3$ putting (4.22) into (4.25), the state equation, results is

$$v(n) = A^2[Av(n-3) + Bx(n-2)] + ABx(n-1) + Bx(n) \tag{4.27}$$

$$v(n) = A^3v(n-3) + A^2Bx(n-2) + ABx(n-1) + Bx(n) \tag{4.28}$$

$$V = A^3z^{-3}V + A^2Bz^{-2}X + ABz^{-1}X + BX. \tag{4.29}$$

For $L = 4$ putting (4.23) into (4.28), the state equation, results is

$$v(n) = A^3[Av(n-4) + Bx(n-3)] + A^2Bx(n-2) + ABx(n-1) + Bx(n) \tag{4.30}$$

$$v(n) = A^4v(n-4) + A^3Bx(n-3) + A^2Bx(n-2) + ABx(n-1) + Bx(n) \tag{4.31}$$

$$V = A^4z^{-4}V + A^3Bz^{-3}X + A^2Bz^{-2}X + ABz^{-1}X + BX. \tag{4.32}$$

Equation (4.32) can be written in the following form for any $L > 1$:

$$V = A^Lz^{-L}V + \sum_{i=0}^{L-1} A^iBz^{-i}X. \tag{4.33}$$

One of the benefits of the state space realization is that if the original filter ($L = 1$) is stable, all higher order realizations are also stable, and furthermore, A^L has improved error properties over A. This advantage of the state space method is discussed in the paper by Lu, Lee, and Messerschmitt [Ref. 4].

D. SECOND ORDER IIR FILTER STATE SPACE REALIZATIONS

From the state space equation of the second order IIR filter in section B and the modified state space equation in section C, we can develop techniques for high speed implementation of second order IIR filters for several values of L. This section shows how to apply these concepts for L = 1, 2, 3, and 4:

1. L = 1

Beginning with equation (4.20),

$$\mathbf{V} = \mathbf{A}z^{-1}\mathbf{V} + \mathbf{B}X \tag{4.34}$$

and substituting for **V**, **A**, and **B** from section B yields:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ 1 & 0 \end{bmatrix} z^{-1} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} a_0 + a_1 z^{-1} + a_2 z^{-2} \\ 0 \end{bmatrix} X. \tag{4.35}$$

Beginning with the output equation, equation (4.17), and referring to Figure 23, we note that the output of the multiplier d is $dz^{-2}X$. This results is a revised output equation of

$$Y = c_1 V_1 + c_2 V_2 + dz^{-2}X. \tag{4.36}$$

Substituting the state equation into the output equation, results in

$$\begin{aligned} Y &= (b_1 c_1 + c_2) z^{-1} V_1 \\ &+ b_2 c_1 z^{-1} V_2 \\ &+ a_0 c_1 X \\ &+ a_1 c_1 z^{-1} X \\ &+ (a_2 c_1 + d) z^{-2} X \end{aligned} \tag{4.37}$$

$$\begin{aligned} Y &= (b_1 c_1 + c_2) z^{-1} Y \\ &+ b_2 c_1 z^{-2} Y \\ &+ a_0 c_1 X \\ &+ a_1 c_1 z^{-1} X \\ &+ (a_2 c_1 + d) z^{-2} X. \end{aligned} \tag{4.38}$$

To demonstrate equality with the Y of the original IIR filter as expressed in equation (3.23) for p = 0, let $c_1 = 1$, $c_2 = 0$, and $d = 0$. Then the output equation is:

$$Y = b_1 z^{-1} Y + b_2 z^{-2} Y + a_0 X + a_1 z^{-1} X + a_2 z^{-2} X. \tag{4.39}$$

By substituting:

$$\begin{aligned} b_1 &= b_{11} \\ b_2 &= b_{12} \\ a_0 &= a_{11} \\ a_1 &= a_{12} \\ a_2 &= a_{13} \end{aligned} \tag{4.40}$$

we can write this equation as:

$$Y = b_{11}z^{-1}Y + b_{12}z^{-2}Y + a_{11}X + a_{12}z^{-1}X + a_{13}z^{-2}X. \tag{4.41}$$

This realization is shown in Figure 23 on page 47.

2. L = 2

Beginning with the state equation (4.26),

$$\mathbf{V} = \mathbf{A}^2 z^{-2}\mathbf{V} + \mathbf{AB}z^{-1}X + \mathbf{B}X \tag{4.42}$$

and substituting for **V**, **A**, and **B** from section B yields:

$$\begin{aligned} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} &= \begin{bmatrix} b_1 & b_2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} b_1 & b_2 \\ 1 & 0 \end{bmatrix} z^{-2} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \\ &+ \begin{bmatrix} b_1 & b_2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_0 + a_1 z^{-1} + a_2 z^{-2} \\ 0 \end{bmatrix} z^{-1}X \\ &+ \begin{bmatrix} a_0 + a_1 z^{-1} + a_2 z^{-2} \\ 0 \end{bmatrix} X \end{aligned} \tag{4.43}$$

$$\begin{aligned} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} &= \begin{bmatrix} b_1^2 + b_2 & b_1 b_2 \\ b_1 & b_2 \end{bmatrix} z^{-2} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \\ &+ \begin{bmatrix} a_0 b_1 + a_1 b_1 z^{-1} + a_2 b_1 z^{-2} \\ a_0 + a_1 z^{-1} + a_2 z^{-2} \end{bmatrix} z^{-1}X \\ &+ \begin{bmatrix} a_0 + a_1 z^{-1} + a_2 z^{-2} \\ 0 \end{bmatrix} X \end{aligned} \tag{4.44}$$

$$\begin{aligned} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} &= \begin{bmatrix} b_1^2 + b_2 & b_1 b_2 \\ b_1 & b_2 \end{bmatrix} z^{-2} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \\ &+ \begin{bmatrix} a_0 + (a_0 b_1 + a_1)z^{-1} + (a_1 b_1 + a_2)z^{-2} + a_2 b_1 z^{-3} \\ a_0 z^{-1} + a_1 z^{-2} + a_2 z^{-3} \end{bmatrix} X. \end{aligned} \tag{4.45}$$

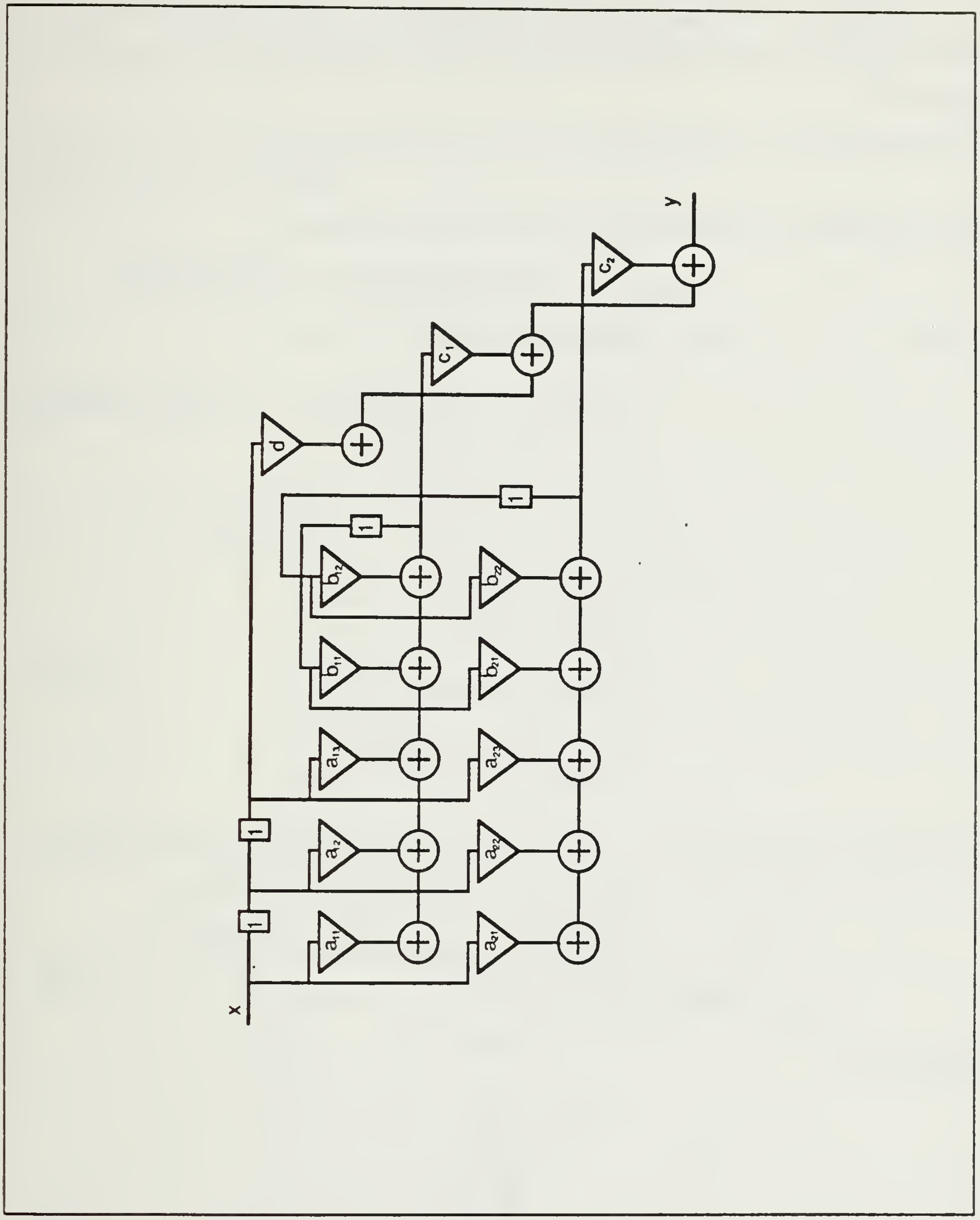

Figure 23. Second Order IIR Filter: L = 1

Beginning with the output equation, equation (4.17), and referring to Figure 24, we note that the output of the multiplier d is $dz^{-3}X$. This results is a revised output equation of

$$Y = c_1 V_1 + c_2 V_2 + dz^{-3} X. \tag{4.46}$$

Substituting the state equation into the output equation, results in

$$\begin{aligned} Y = &\left[(b_1^2 + b_2)c_1 + b_1 c_2\right] z^{-2} V_1 \\ &+ [b_1 b_2 c_1 + b_2 c_2] z^{-2} V_2 \\ &+ a_0 c_1 X \\ &+ [(a_0 b_1 + a_1)c_1 + a_0 c_2] z^{-1} X \\ &+ (a_1 b_1 c_1 + a_2 c_1 + a_1 c_2) z^{-2} X \\ &+ (a_2 b_1 c_1 + a_2 c_2 + d) z^{-3} X \end{aligned} \tag{4.47}$$

$$\begin{aligned} Y = &\left[(b_1^2 + b_2)c_1 + b_1 c_2\right] z^{-2} Y \\ &+ [b_1 b_2 c_1 + b_2 c_2] z^{-3} Y \\ &+ a_0 c_1 X \\ &+ [(a_0 b_1 + a_1)c_1 + a_0 c_2] z^{-1} X \\ &+ (a_1 b_1 c_1 + a_2 c_1 + a_1 c_2) z^{-2} X \\ &+ (a_2 b_1 c_1 + a_2 c_2 + d) z^{-3} X. \end{aligned} \tag{4.48}$$

To demonstrate equality with the Y of the augmented IIR filter as expressed in equation (3.34) for p = 1, let $c_1 = 1$, $c_2 = 0$, and $d = 0$. Then the output equation is:

$$\begin{aligned} Y = &(b_1^2 + b_2) z^{-2} Y + b_1 b_2 z^{-3} Y + a_0 X + (a_0 b_1 + a_1) z^{-1} X \\ &+ (a_1 b_1 + a_2) z^{-2} X + a_2 b_1 z^{-3} X. \end{aligned} \tag{4.49}$$

By substituting:

$$\begin{aligned} b_1^2 + b_2 &= b_{11} \\ b_1 b_2 &= b_{12} \\ a_0 &= a_{11} \\ a_0 b_1 + a_1 &= a_{12} \\ a_1 b_1 + a_2 &= a_{13} \\ a_2 b_1 &= a_{14} \end{aligned} \tag{4.50}$$

we can write this equation as:

$$Y = b_{11}z^{-2}Y + b_{12}z^{-3}Y + a_{11}X + a_{12}z^{-1}X + a_{13}z^{-2}X + a_{14}z^{-3}X. \tag{4.51}$$

This is the same as the equation (3.34) for p= 1 found on page 15. This realization is shown in Figure 24 on page 50.

3. L = 3

Beginning with the state equation (4.29),

$$\mathbf{V} = \mathbf{A}^3 z^{-3}\mathbf{V} + \mathbf{A}^2\mathbf{B}z^{-2}X + \mathbf{AB}z^{-1}X + \mathbf{B}X \tag{4.52}$$

and substituting for **V**, **A**, and **B** from section B yields:

$$\begin{aligned}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = &\begin{bmatrix} b_1^2 + b_2 & b_1 b_2 \\ b_1 & b_2 \end{bmatrix}\begin{bmatrix} b_1 & b_2 \\ 1 & 0 \end{bmatrix} z^{-3} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \\ &+ \begin{bmatrix} b_1^2 + b_2 & b_1 b_2 \\ b_1 & b_2 \end{bmatrix}\begin{bmatrix} a_0 + a_1 z^{-1} + a_2 z^{-2} \\ 0 \end{bmatrix} z^{-2}X \\ &+ \begin{bmatrix} b_1 & b_2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_0 + a_1 z^{-1} + a_2 z^{-2} \\ 0 \end{bmatrix} z^{-1}X \\ &+ \begin{bmatrix} a_0 + a_1 z^{-1} + a_2 z^{-2} \\ 0 \end{bmatrix} X \end{aligned} \tag{4.53}$$

$$\begin{aligned}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = &\begin{bmatrix} b_1(b_1^2 + 2b_2) & b_2(b_1^2 + b_2) \\ b_1^2 + b_2 & b_1 b_2 \end{bmatrix} z^{-3} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \\ &+ \begin{bmatrix} a_0(b_1^2 + b_2) + a_1(b_1^2 + b_2)z^{-1} + a_2(b_1^2 + b_2)z^{-2} \\ a_0 b_1 + a_1 b_1 z^{-1} + a_2 b_1 z^{-2} \end{bmatrix} z^{-2}X \\ &+ \begin{bmatrix} a_0 b_1 + a_1 b_1 z^{-1} + a_2 b_1 z^{-2} \\ a_0 + a_1 z^{-1} + a_2 z^{-2} \end{bmatrix} z^{-1}X \\ &+ \begin{bmatrix} a_0 + a_1 z^{-1} + a_2 z^{-2} \\ 0 \end{bmatrix} X \end{aligned} \tag{4.54}$$

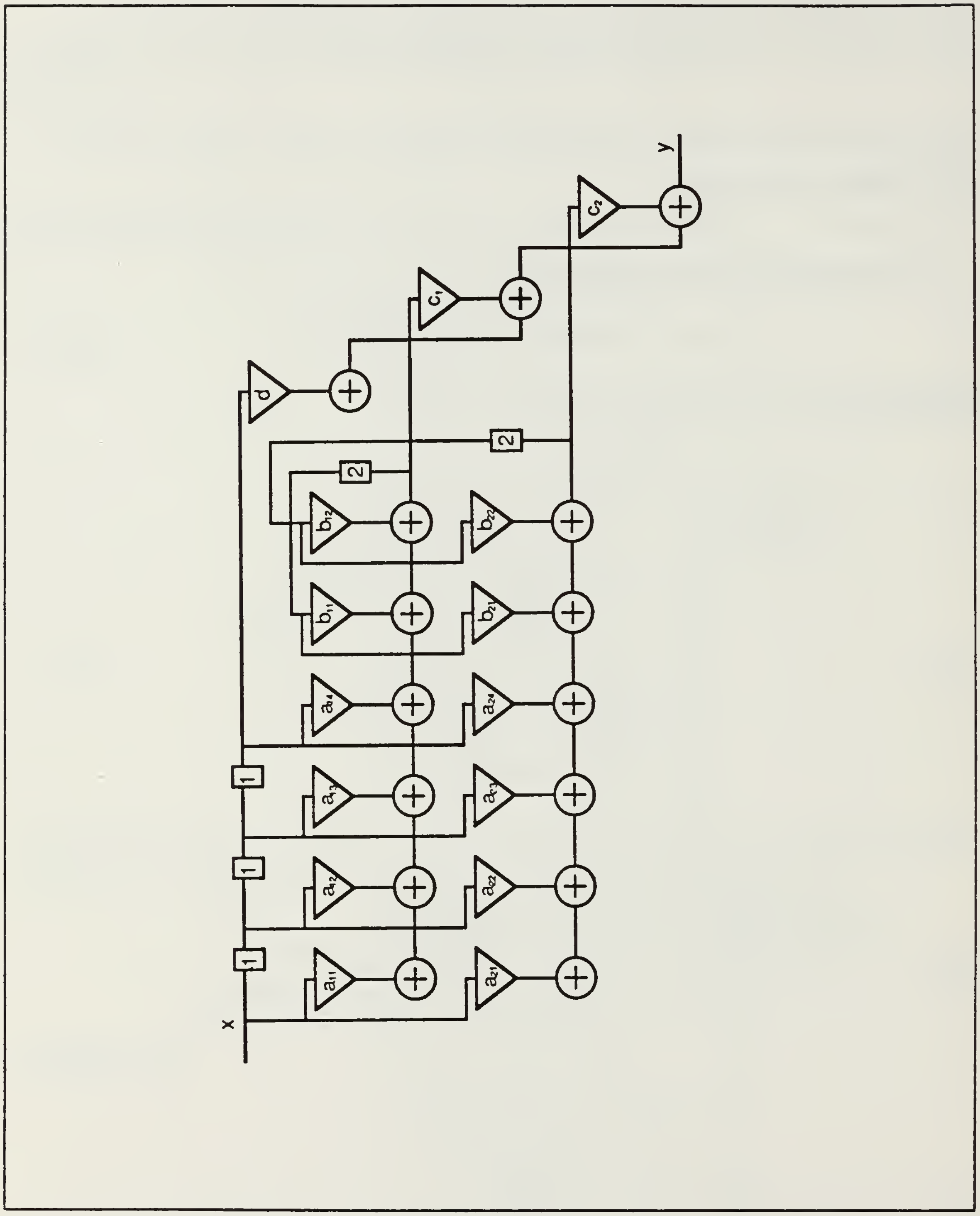

Figure 24. The Augmented IIR Filter: L = 2

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} b_1(b_1^2 + 2b_2) & b_2(b_1^2 + b_2) \\ b_1^2 + b_2 & b_1 b_2 \end{bmatrix} z^{-3} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$+ \begin{bmatrix} a_0 + (a_0 b_1 + a_1)z^{-1} + (a_0 b_1^2 + a_0 b_2 + a_1 b_1 + a_2)z^{-2} \\ + (a_1 b_1^2 + a_1 b_2 + a_2 b_1)z^{-3} + (a_2 b_1^2 + a_2 b_2)z^{-4} \\ a_0 z^{-1} + (a_0 b_1 + a_1)z^{-2} + (a_1 b_1 + a_2)z^{-3} + a_2 b_1 z^{-4} \end{bmatrix} X. \tag{4.55}$$

Beginning with the output equation, equation (4.17), and referring to Figure 25, we note that output of the multiplier d is $dz^{-4}X$. This result is a revised output equation of

$$Y = c_1 V_1 + c_2 V_2 + dz^{-4} X. \tag{4.56}$$

Substituting the state equation into the output equation, results in

$$\begin{aligned} Y = {} & \left[(b_1^3 + 2b_1 b_2)c_1 + (b_1^2 + b_2)c_2\right]z^{-3} V_1 \\ & + \left[(b_1^2 b_2 + b_2^2)c_1 + b_1 b_2 c_2\right]z^{-3} V_2 \\ & + a_0 c_1 X \\ & + \left[(a_0 b_1 + a_1)c_1 + a_0 c_2\right]z^{-1} X \\ & + \left[(a_0 b_1^2 + a_0 b_2 + a_1 b_1 + a_2)c_1 + (a_0 b_1 + a_1)c_2\right]z^{-2} X \\ & + \left[(a_1 b_1^2 + a_1 b_2 + a_2 b_1)c_1 + (a_1 b_1 + a_2)c_2\right]z^{-3} X \\ & + \left[(a_2 b_1^2 + a_2 b_2)c_1 + a_2 b_1 c_2 + d\right]z^{-4} X \end{aligned} \tag{4.57}$$

$$\begin{aligned} Y = {} & \left[(b_1^3 + 2b_1 b_2)c_1 + (b_1^2 + b_2)c_2\right]z^{-3} Y \\ & + \left[(b_1^2 b_2 + b_2^2)c_1 + b_1 b_2 c_2\right]z^{-4} Y \\ & + a_0 c_1 X \\ & + \left[(a_0 b_1 + a_1)c_1 + a_0 c_2\right]z^{-1} X \\ & + \left[(a_0 b_1^2 + a_0 b_2 + a_1 b_1 + a_2)c_1 + (a_0 b_1 + a_1)c_2\right]z^{-2} X \\ & + \left[(a_1 b_1^2 + a_1 b_2 + a_2 b_1)c_1 + (a_1 b_1 + a_2)c_2\right]z^{-3} X \\ & + \left[(a_2 b_1^2 + a_2 b_2)c_1 + a_2 b_1 c_2 + d\right]z^{-4} X. \end{aligned} \tag{4.58}$$

To demonstrate equality with the Y of the augmented IIR filter as expressed in equation (3.45) for p = 2, let $c_1 = 1$, $c_2 = 0$, and $d = 0$. Then the output equation is:

$$Y = (b_1^3 + 2b_1b_2)z^{-3}Y + (b_1^2b_2 + b_2^2)z^{-4}Y + a_0X + (a_0b_1 + a_1)z^{-1}X$$
$$+ \left[a_0(b_1^2 + b_2) + a_1b_1 + a_2\right]z^{-2}X + \left[a_1(b_1^2 + b_2) + a_2b_1\right]z^{-3}X \qquad (4.59)$$
$$+ a_2(b_1^2 + b_2)z^{-4}X.$$

By substituting:

$$\begin{aligned} b_1^3 + 2b_1b_2 &= b_{11} \\ b_1^2b_2 + b_2^2 &= b_{12} \\ a_0 &= a_{11} \\ a_0b_1 + a_1 &= a_{12} \\ a_0(b_1^2 + b_2) + a_1b_1 + a_2 &= a_{13} \\ a_1(b_1^2 + b_2) + a_2b_1 &= a_{14} \\ a_2(b_1^2 + b_2) &= a_{15} \end{aligned} \qquad (4.60)$$

we can write this equation as:

$$\begin{aligned} Y = b_{11}z^{-3}Y + b_{12}z^{-4}Y + a_{11}X + a_{12}z^{-1}X \\ + a_{13}z^{-2}X + a_{14}z^{-3}X + a_{15}z^{-4}X. \end{aligned} \qquad (4.61)$$

This is the same as the equation (3.45) for p = 2 found on page 17. This realization is shown in Figure 25 on page 53.

4. L = 4

Beginning with equation (4.32),

$$\mathbf{V} = \mathbf{A}^4z^{-4}\mathbf{V} + \mathbf{A}^3\mathbf{B}z^{-3}X + \mathbf{A}^2\mathbf{B}z^{-2}X + \mathbf{AB}z^{-1}X + \mathbf{B}X \qquad (4.62)$$

and substituting for **V**, **A**, and **B** from section B yields:

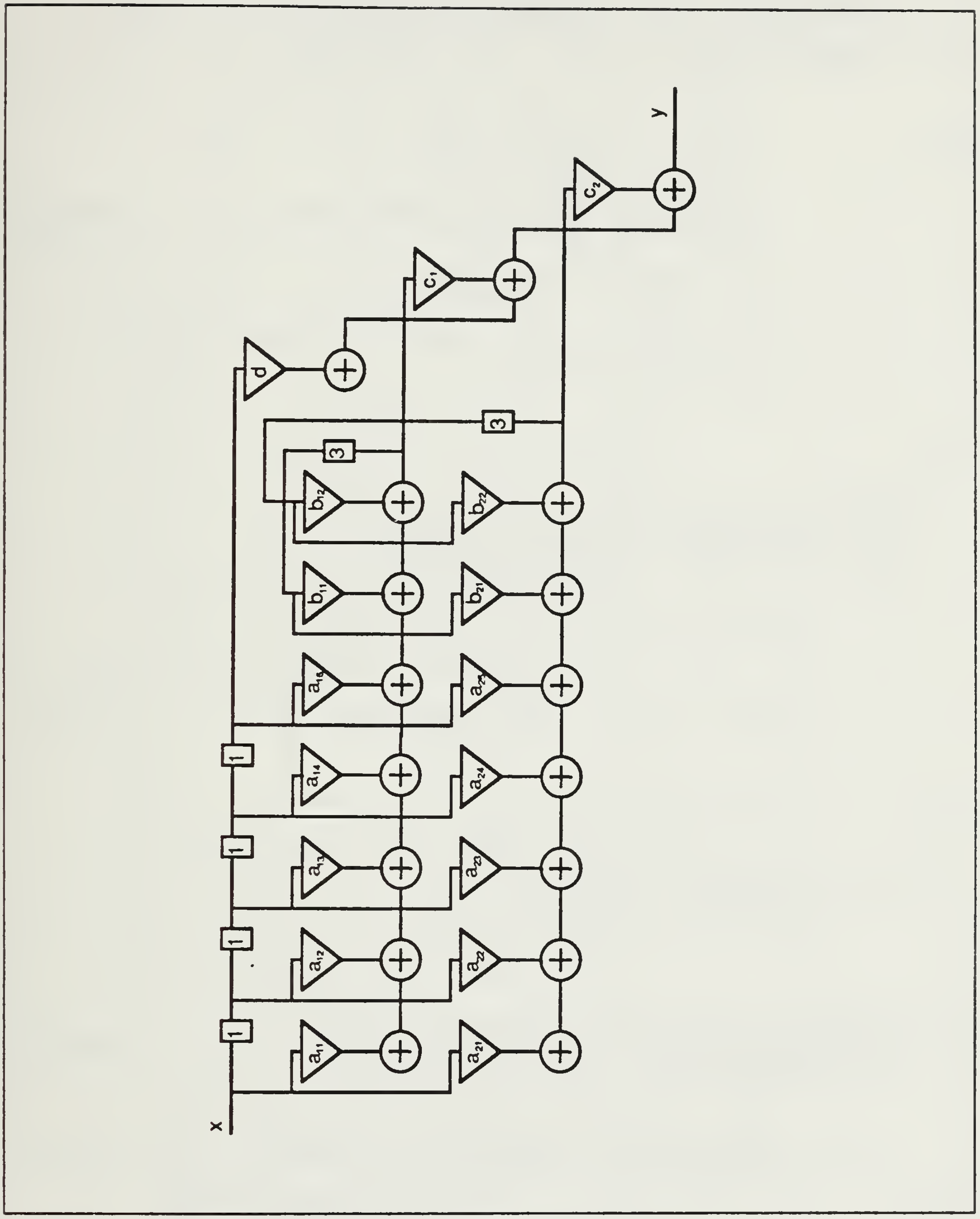

Figure 25. The Augmented IIR Filter: L = 3

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} b_1(b_1^2+2b_2) & b_2(b_1^2+b_2) \\ b_1^2+b_2 & b_1b_2 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ 1 & 0 \end{bmatrix} z^{-4} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$
$$+ \begin{bmatrix} b_1(b_1^2+2b_2) & b_2(b_1^2+b_2) \\ b_1^2+b_2 & b_1b_2 \end{bmatrix} \begin{bmatrix} a_0+a_1z^{-1}+a_2z^{-2} \\ 0 \end{bmatrix} z^{-3}X$$
$$+ \begin{bmatrix} b_1^2+b_2 & b_1b_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} a_0+a_1z^{-1}+a_2z^{-2} \\ 0 \end{bmatrix} z^{-2}X \tag{4.63}$$
$$+ \begin{bmatrix} b_1 & b_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_0+a_1z^{-1}+a_2z^{-2} \\ 0 \end{bmatrix} z^{-1}X$$
$$+ \begin{bmatrix} a_0+a_1z^{-1}+a_2z^{-2} \\ 0 \end{bmatrix} X$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} b_1^4+3b_1^2b_2+b_2^2 & b_1b_2(b_1^2+2b_2) \\ b_1(b_1^2+2b_2) & b_2(b_1^2+b_2) \end{bmatrix} z^{-4} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$
$$+ \begin{bmatrix} a_0b_1(b_1^2+2b_2)+a_1b_1(b_1^2+2b_2)z^{-1}+a_2b_1(b_1^2+2b_2)z^{-2} \\ a_0(b_1^2+b_2)+a_1(b_1^2+b_2)z^{-1}+a_2(b_1^2+b_2)z^{-2} \end{bmatrix} z^{-3}X$$
$$+ \begin{bmatrix} a_0(b_1^2+b_2)+a_1(b_1^2+b_2)z^{-1}+a_2(b_1^2+b_2)z^{-2} \\ a_0b_1+a_1b_1z^{-1}+a_2b_1z^{-2} \end{bmatrix} z^{-2}X \tag{4.64}$$
$$+ \begin{bmatrix} a_0b_1+a_1b_1z^{-1}+a_2b_1z^{-2} \\ a_0+a_1z^{-1}+a_2z^{-2} \end{bmatrix} z^{-1}X$$
$$+ \begin{bmatrix} a_0+a_1z^{-1}+a_2z^{-2} \\ 0 \end{bmatrix} X$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} b_1^4+3b_1^2b_2+b_2^2 & b_1b_2(b_1^2+2b_2) \\ b_1(b_1^2+2b_2) & b_2(b_1^2+b_2) \end{bmatrix} z^{-4} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$
$$+ \begin{bmatrix} a_0+(a_0b_1+a_1)z^{-1}+(a_0b_1^2+a_0b_2+a_1b_1+a_2)z^{-2} \\ +(a_0b_1^3+2a_0b_1b_2+a_1b_1^2+a_1b_2+a_2b_1)z^{-3} \\ +(a_1b_1^3+2a_1b_1b_2+a_2b_2)z^{-4}+(a_2b_1^3+2a_2b_1b_2)z^{-5} \\ a_0z^{-1}+(a_0b_1+a_1)z^{-2}+(a_0b_1^2+a_0b_2+a_1b_1+a_2)z^{-3} \\ +(a_1b_1^2+a_1b_2+a_2b_1)z^{-4}+(a_2b_1^2+a_2b_2)z^{-5} \end{bmatrix} X. \tag{4.65}$$

Beginning with the output equation, equation (4.17), and referring to Figure 26, we note that the output of the multiplier d is $dz^{-5}X$. This results is a revised output equation of

$$Y = c_1 V_1 + c_2 V_2 + dz^{-5}X. \tag{4.66}$$

Substituting the state equation into the output equation, results in

$$\begin{aligned}
Y = {} & \left[(b_1^4 + 3b_1^2 b_2 + b_2^2)c_1 + (b_1^3 + 2b_1 b_2)c_2\right]z^{-4}V_1 \\
& + \left[(b_1^3 b_2 + 2b_1 b_2^2)c_1 + (b_1^2 b_2 + b_2^2)c_2\right]z^{-4}V_2 \\
& + a_0 c_1 X \\
& + \left[(a_0 b_1 + a_1)c_1 + a_0 c_2\right]z^{-1}X \\
& + \left[(a_0 b_1^2 + a_0 b_2 + a_1 b_1 + a_2)c_1 + (a_0 b_1 + a_1)c_2\right]z^{-2}X \\
& + \left[(a_0 b_1^3 + 2a_0 b_1 b_2 + a_1 b_1^2 + a_1 b_2 + a_2 b_1)c_1 + (a_0 b_1^2 + a_0 b_2 + a_1 b_1 + a_2)c_2\right]z^{-3}X \\
& + \left[(a_1 b_1^3 + 2a_1 b_1 b_2 + a_2 b_1^2 + a_2 b_2)c_1 + (a_1 b_1^2 + a_1 b_2 + a_2 b_1)c_2\right]z^{-4}X \\
& + \left[(a_2 b_1^3 + 2a_2 b_1 b_2)c_1 + (a_2 b_1^2 + a_2 b_2)c_2 + d\right]z^{-5}X
\end{aligned} \tag{4.67}$$

$$\begin{aligned}
Y = {} & \left[(b_1^4 + 3b_1^2 b_2 + b_2^2)c_1 + (b_1^3 + 2b_1 b_2)c_2\right]z^{-4}Y \\
& + \left[(b_1^3 b_2 + 2b_1 b_2^2)c_1 + (b_1^2 b_2 + b_2^2)c_2\right]z^{-5}Y \\
& + a_0 c_1 X \\
& + \left[(a_0 b_1 + a_1)c_1 + a_0 c_2\right]z^{-1}X \\
& + \left[(a_0 b_1^2 + a_0 b_2 + a_1 b_1 + a_2)c_1 + (a_0 b_1 + a_1)c_2\right]z^{-2}X \\
& + \left[(a_0 b_1^3 + 2a_0 b_1 b_2 + a_1 b_1^2 + a_1 b_2 + a_2 b_1)c_1 + (a_0 b_1^2 + a_0 b_2 + a_1 b_1 + a_2)c_2\right]z^{-3}X \\
& + \left[(a_1 b_1^3 + 2a_1 b_1 b_2 + a_2 b_1^2 + a_2 b_2)c_1 + (a_1 b_1^2 + a_1 b_2 + a_2 b_1)c_2\right]z^{-4}X \\
& + \left[(a_2 b_1^3 + 2a_2 b_1 b_2)c_1 + (a_2 b_1^2 + a_2 b_2)c_2 + d\right]z^{-5}X.
\end{aligned} \tag{4.68}$$

To demonstrate equality with the Y of the augmented IIR filter as expressed in equation (3.56) for p = 3, let $c_1 = 1$, $c_2 = 0$, and $d = 0$. Then the output equation is:

$$\begin{aligned}
Y = {} & (b_1^4 + 3b_1^2 b_2 + b_2^2)z^{-4}Y + (b_1^3 b_2 + 2b_1 b_2^2)z^{-5}Y + a_0 X + (a_0 b_1 + a_1)z^{-1}X \\
& + \left[a_0(b_1^2 + b_2) + a_1 b_1 + a_2\right]z^{-2}X \\
& + \left[a_0(b_1^3 + 2b_1 b_2) + a_1(b_1^2 + b_2) + a_2 b_1\right]z^{-3}X \\
& + \left[a_1(b_1^3 + 2b_1 b_2) + a_2(b_1^2 + b_2)\right]z^{-4}X + a_2(b_1^3 + 2b_1 b_2)z^{-5}X.
\end{aligned} \tag{4.69}$$

By substituting:

$$
\begin{aligned}
b_1^4 + 3b_1^2 b_2 + b_2^2 &= b_{11} \\
b_1^3 b_2 + 2b_1 b_2^2 &= b_{12} \\
a_0 &= a_{11} \\
a_0 b_1 + a_1 &= a_{12} \\
a_0(b_1^2 + b_2) + a_1 b_1 + a_2 &= a_{13} \\
a_0(b_1^3 + 2b_1 b_2) + a_1(b_1^2 + b_2) + a_2 b_1 &= a_{14} \\
a_1(b_1^3 + 2b_1 b_2) + a_2(b_1^2 + b_2) &= a_{15} \\
a_2(b_1^3 + 2b_1 b_2) &= a_{16}
\end{aligned}
\tag{4.70}
$$

we can write this equation as:

$$
\begin{aligned}
Y = b_{11}z^{-4}Y + b_{12}z^{-5}Y + a_{11}X + a_{12}z^{-1}X \\
+ a_{13}z^{-2}X + a_{14}z^{-3}X \\
+ a_{15}z^{-4}X + a_{16}z^{-5}X.
\end{aligned}
\tag{4.71}
$$

This is the same as the equation (3.56) for p = 3 on page 20. This realization is shown in Figure 26 on page 57.

From equation (4.33) we can see that the feedback portion of the realization must have a delay of L stages in any loop. The general realization of that portion is shown in Figure 27 on page 58.

E. PIPELINED IMPLEMENTATION

All of the IIR filters in section D can be developed into high-speed IIR filters by applying pipeline techniques. We use Ka-stage pipeline add units and Km-stage pipeline multipliers to increase the speed of the filters. The Km stages are involved in the coefficient multiplies and the Ka stages are involved in the adds. The net result is a (Km + Ka)-stage pipeline multiply-add unit. The equations for the filter outputs for different L's are given below:

1. L = 1

The equation is

$$
z^{-(8Ka+2Km)}Y = z^{-(8Ka+2Km)}\left[\begin{array}{l} b_{11}z^{-(2Ka+Km)}Y + b_{12}z^{-(2Ka+Km+1)}Y \\ + a_{11}X + a_{12}z^{-1}X + a_{13}z^{-2}X \end{array}\right].
\tag{4.72}
$$

This realization is shown in Figure 28 on page 59.

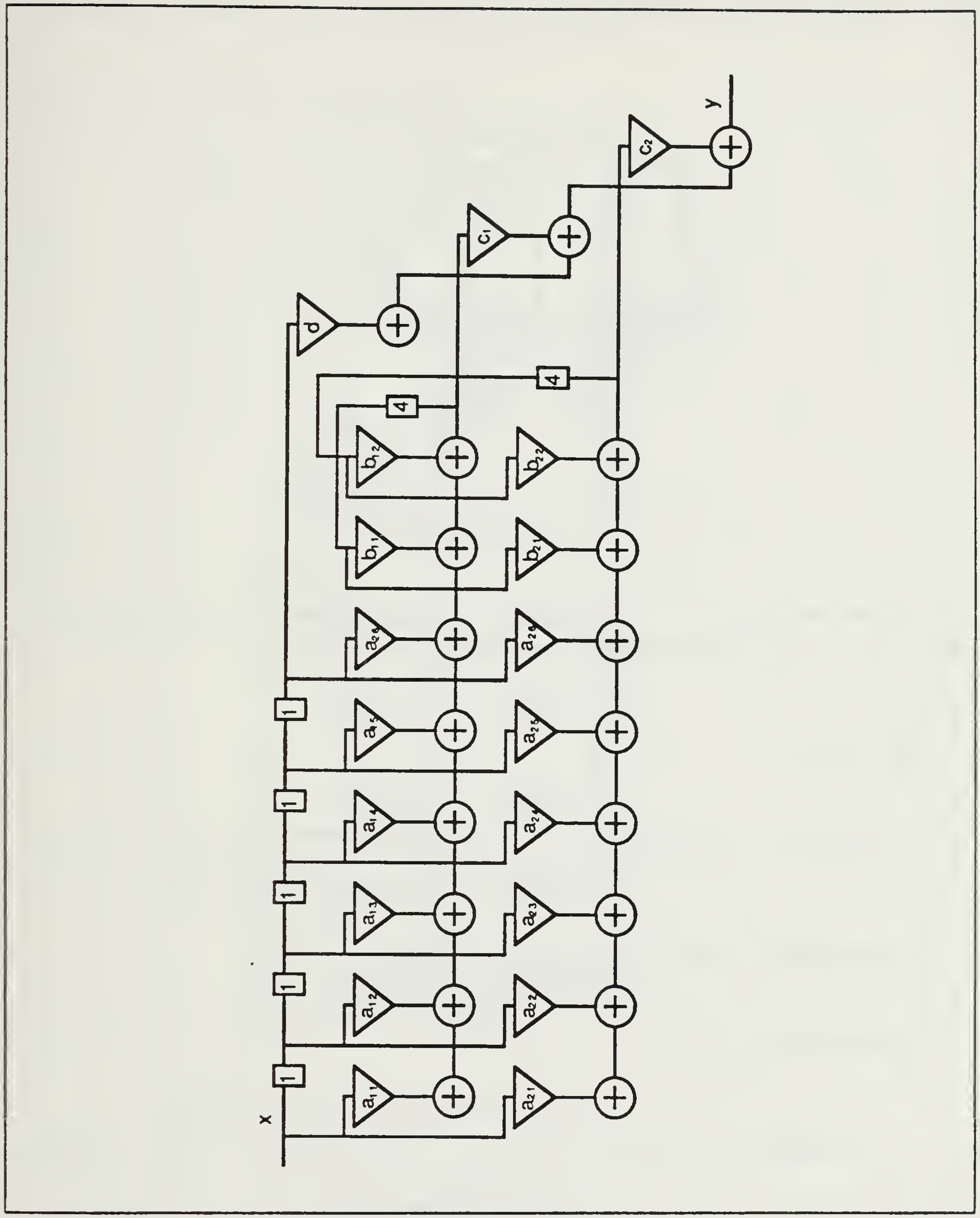

Figure 26. The Augmented IIR Filter: $L = 4$

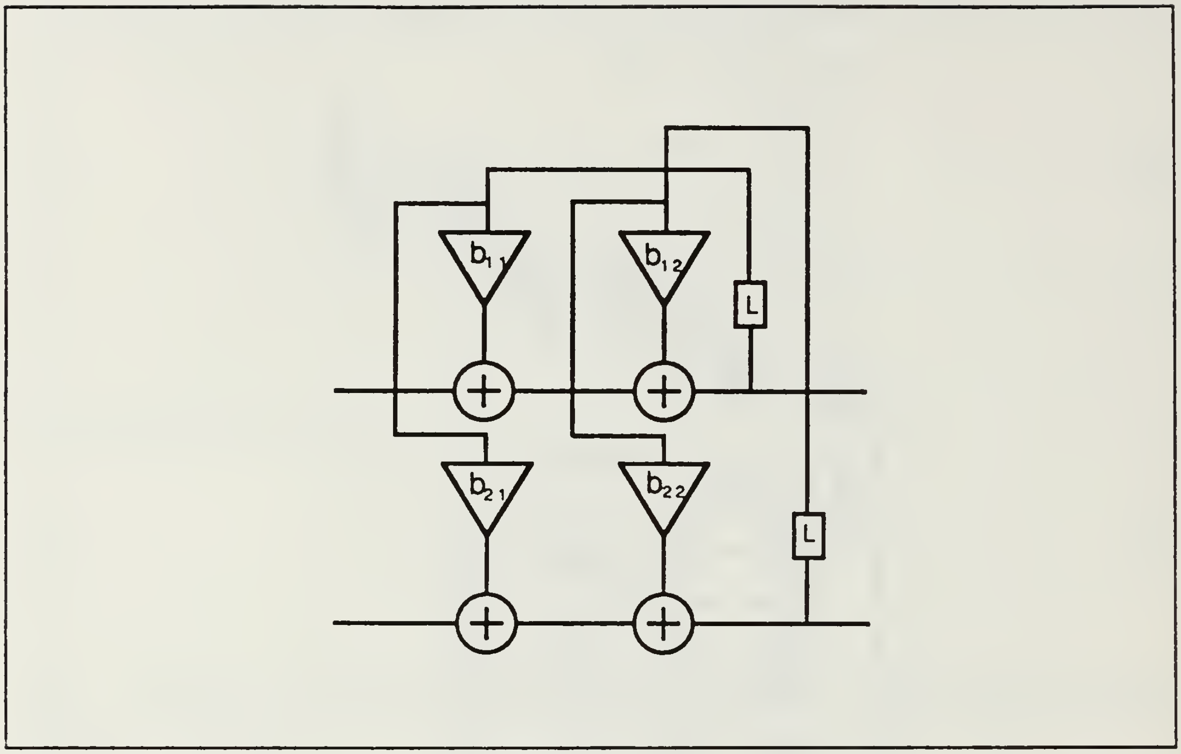

Figure 27. Heart of the Pipeline Recursive Filter Realization

2. L = 2

The equation is

$$z^{-(9Ka+2Km)}Y = z^{-(9Ka+2Km)}\begin{bmatrix} b_{11}z^{-(2Ka+Km)}Y + b_{12}z^{-(2Ka+Km+1)}Y \\ + a_{11}X + a_{12}z^{-1}X + a_{13}z^{-2}X + a_{14}z^{-3}X \end{bmatrix}. \tag{4.73}$$

This realization is shown in Figure 29 on page 60.

3. L = 3

The equation is

$$z^{-(10Ka+2Km)}Y = z^{-(10Ka+2Km)}\begin{bmatrix} b_{11}z^{-(2Ka+Km)}Y + b_{12}z^{-(2Ka+Km+1)}Y \\ + a_{11}X + a_{12}z^{-1}X + a_{13}z^{-2}X \\ + a_{14}z^{-3}X + a_{15}z^{-4}X \end{bmatrix}. \tag{4.74}$$

This realization is shown in Figure 30 on page 61.

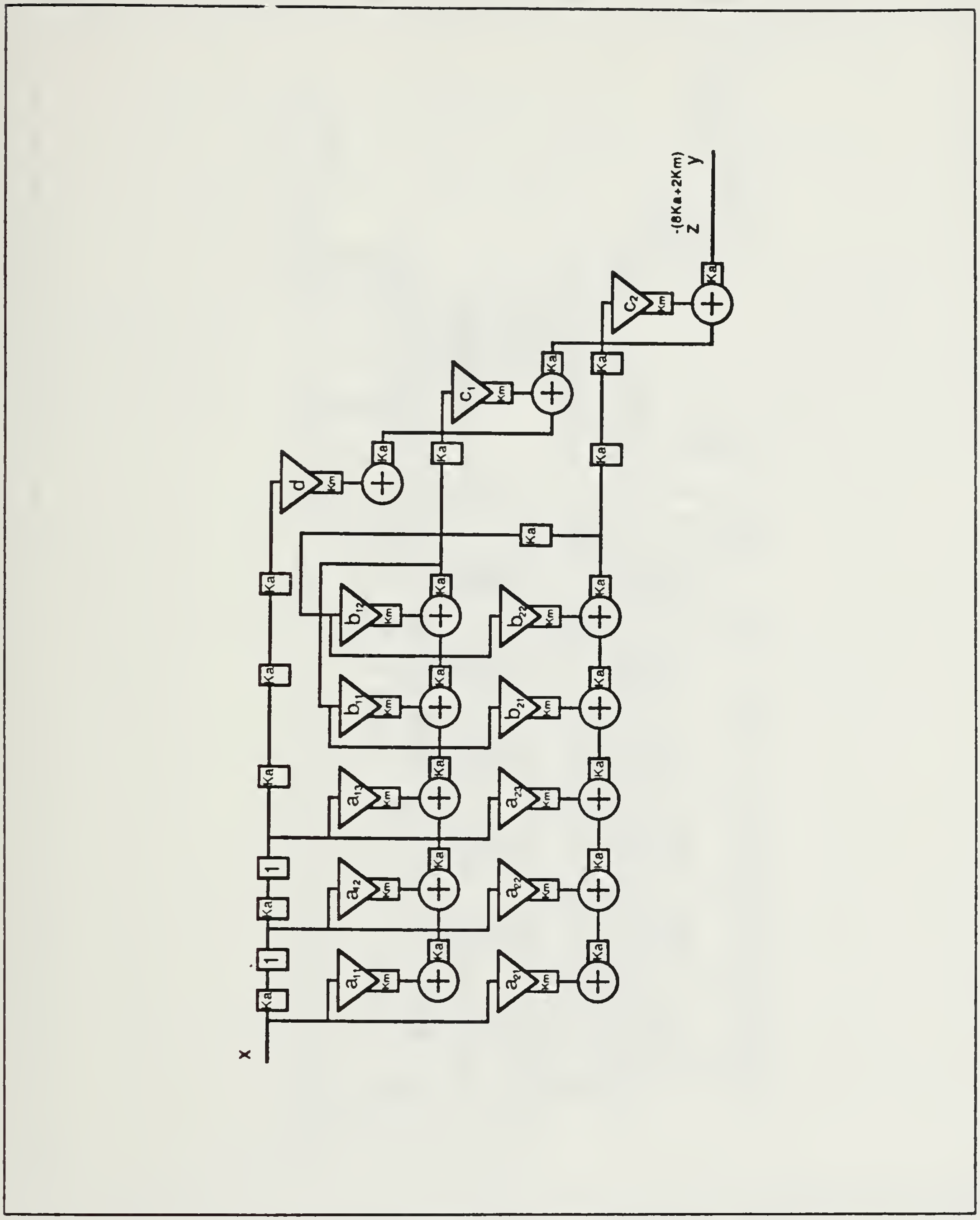

Figure 28. Second Order IIR Filter: L = 1 with Ka, Km

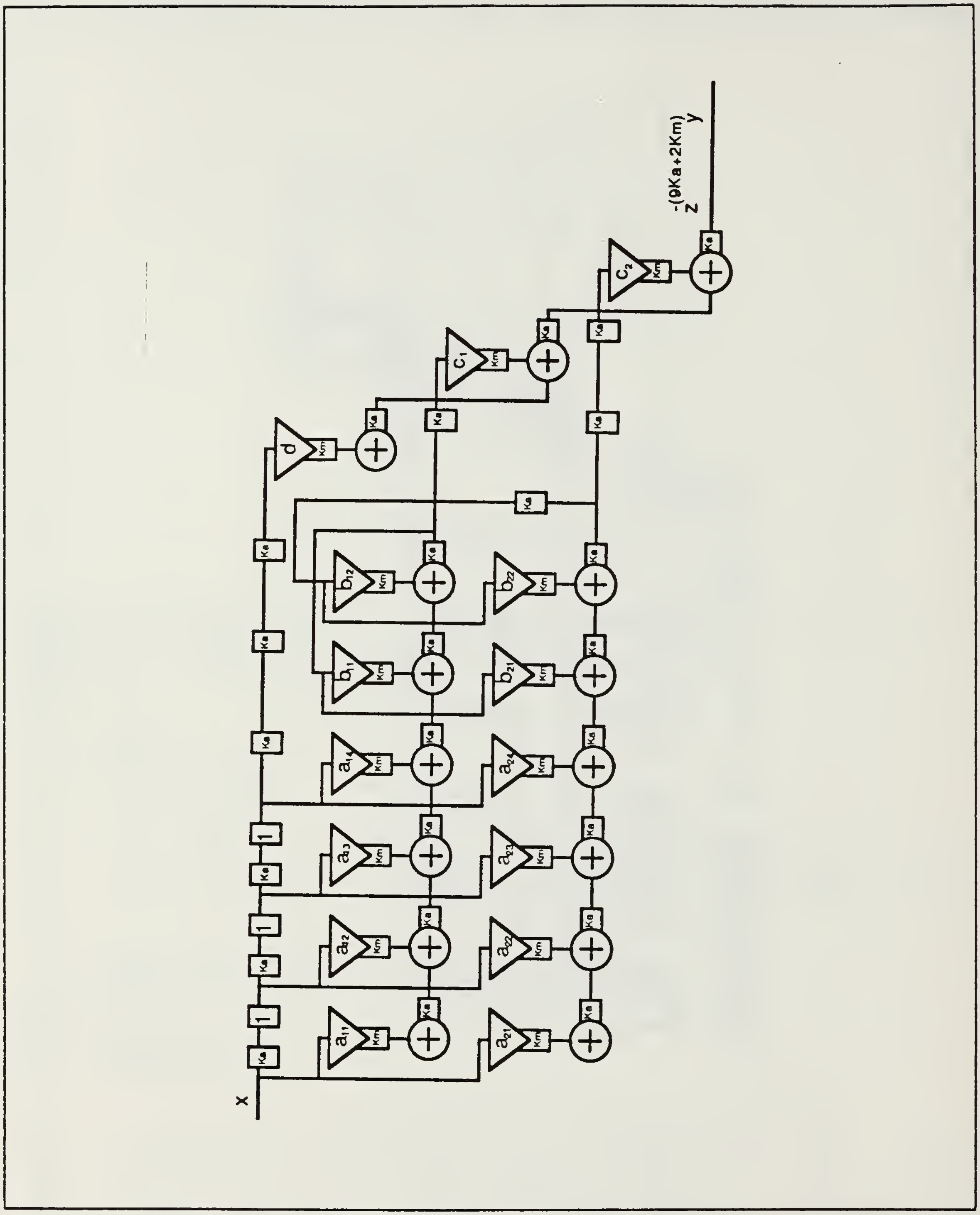

Figure 29. The Augmented IIR Filter: L = 2 with Ka, Km

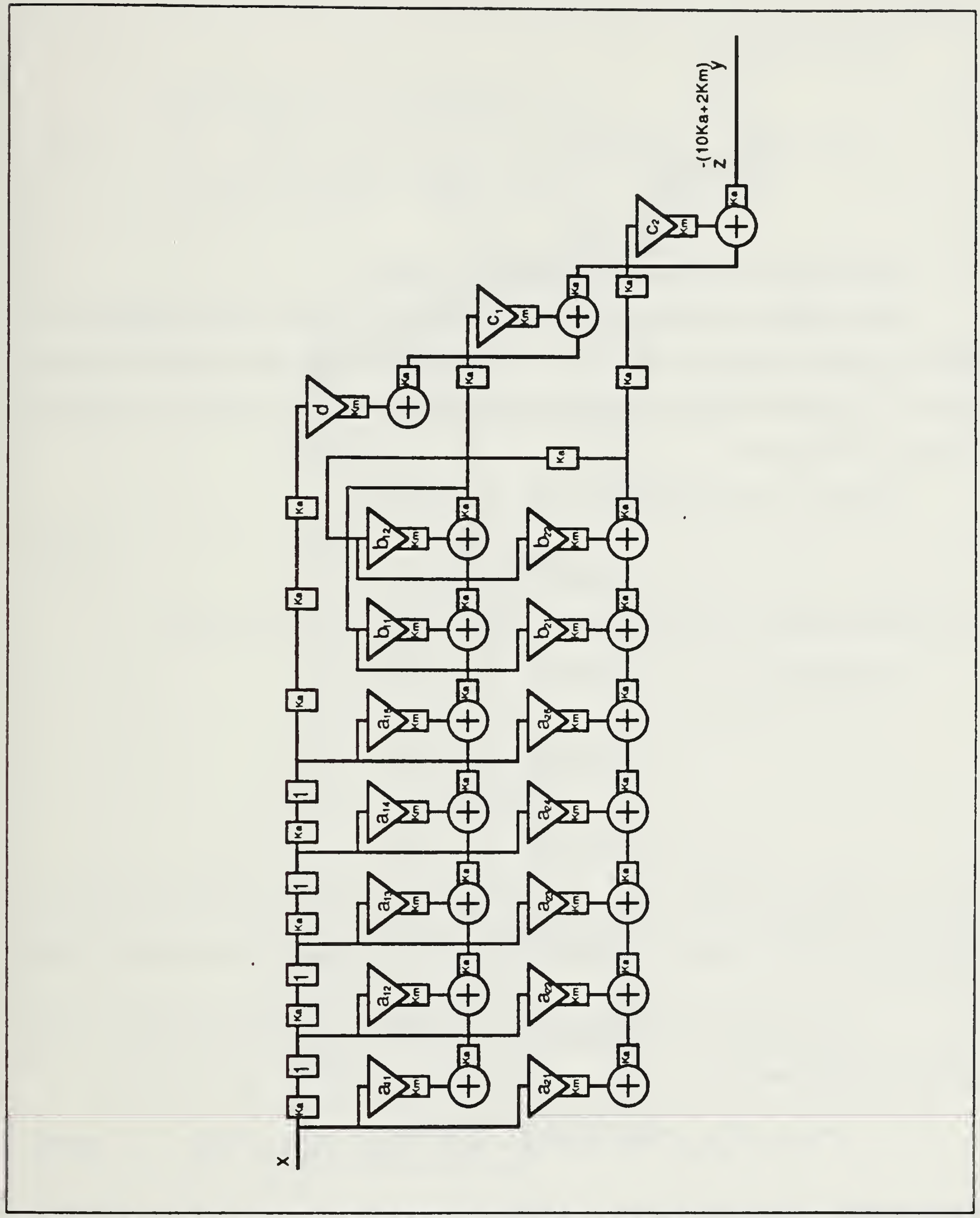

Figure 30. The Augmented IIR Filter: L = 3 with Ka, Km

4. L = 4

The equation is

$$z^{-(11Ka+2Km)}Y = z^{-(11Ka+2Km)}\begin{bmatrix} b_{11}z^{-(2Ka+Km)}Y + b_{12}z^{-(2Ka+Km+1)}Y \\ + a_{11}X + a_{12}z^{-1}X + a_{13}z^{-2}X \\ + a_{14}z^{-3}X + a_{15}z^{-4}X + a_{16}z^{-5}X \end{bmatrix}. \tag{4.75}$$

This realization is shown in Figure 31 on page 63.

From analysis of the feed back loop portion of the state space realization, as shown in Figure 27 and equation (4.33), we have seen that the delay around any loop must be L. A general realization for the feedback portion of the pipeline case is shown in Figure 32 on page 64.

Therefore we can see that

$$L = 2Ka + Km. \tag{4.76}$$

The following examples show the analysis of the cases for L = 1, 2, 3 and 4.

5. **Example for L = 1:**

a. *Ka = 0, Km = 1*

$$\begin{aligned} L &= 2Ka + Km \\ &= 2(0) + 1 \\ &= 1 \end{aligned}$$

The only integer, non-negative values for Ka and Km which solve equation (4.76) are $Ka = 0$, $Km = 1$. This means we cannot use pipeline modules with longer delays than $Ka = 0$, $Km = 1$.

This realizes the original IIR filter for L = 1, but with an additional 2 delays and the equation is

$$z^{-2}Y = z^{-2}\left[b_{11}z^{-1}Y + b_{12}z^{-2}Y + a_{11}X + a_{12}z^{-1}X + a_{13}z^{-2}X\right]. \tag{4.77}$$

This realization is shown in Figure 33 on page 65.

6. **Example for L = 2:**

The only integer, non-negative values for Ka and Km which solve equation (4.76) are $Ka = 0$, $Km = 2$ or $Ka = 1$, $Km = 0$.

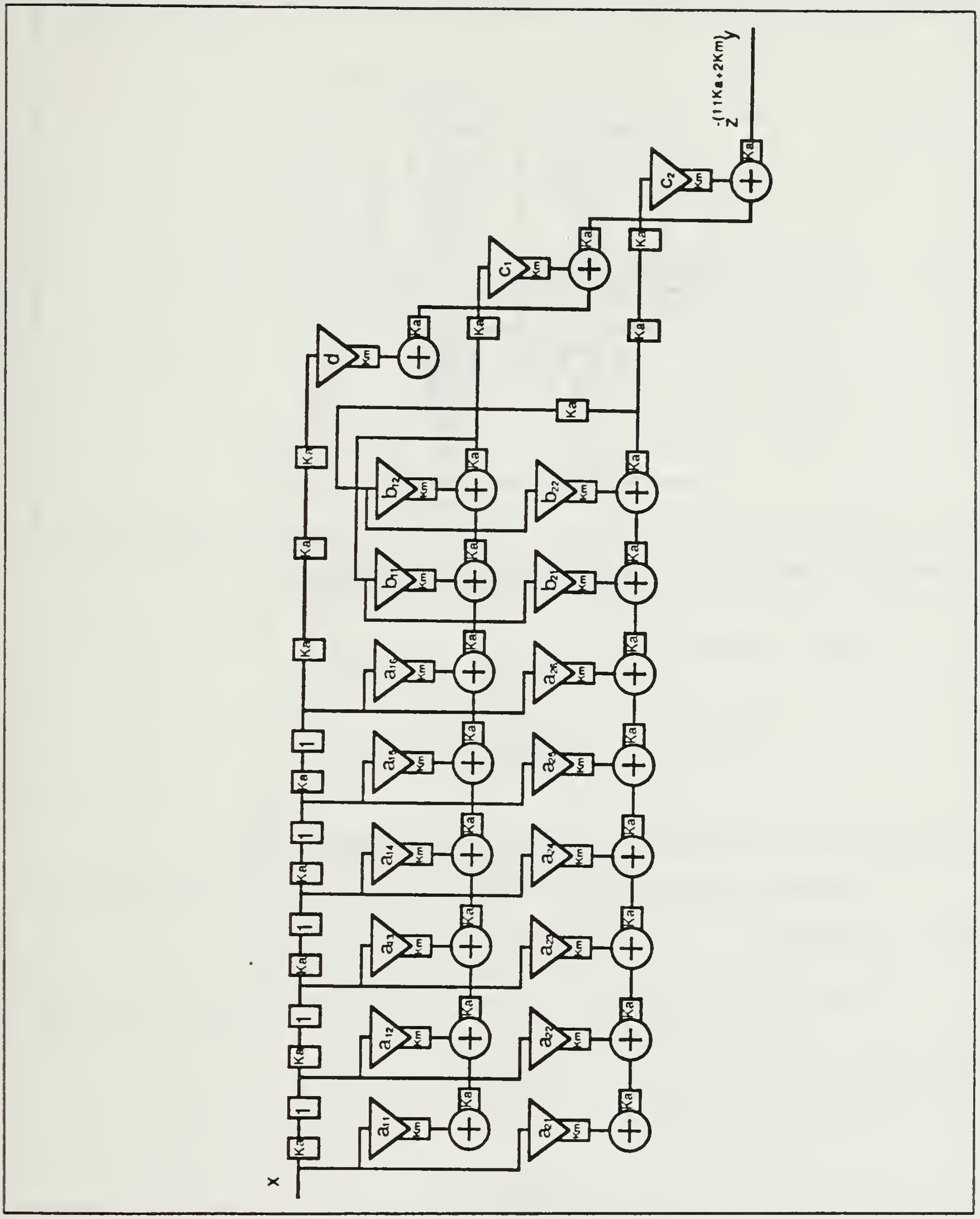

Figure 31. The Augmented IIR Filter: L = 4 with Ka, Km

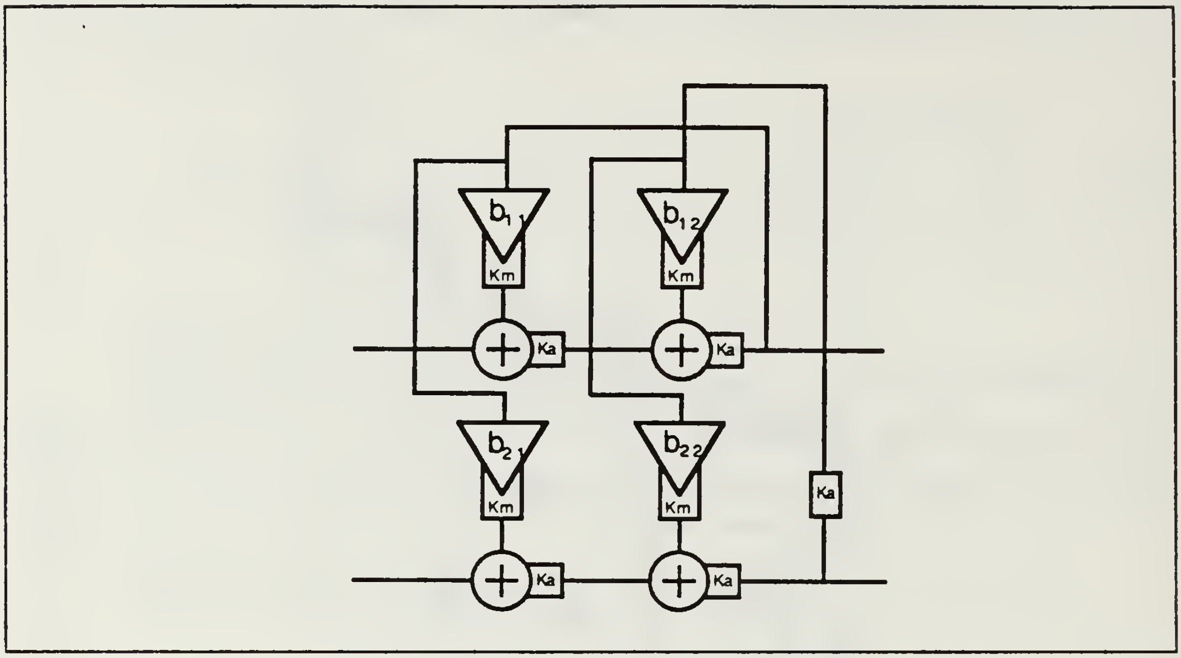

Figure 32. Heart of More General Pipeline Filter Realization

a. $Ka = 0, Km = 2$

$$\begin{aligned} L &= 2Ka + Km \\ &= 2(0) + 2 \\ &= 2 \end{aligned}$$

For these values, we use pipeline modules with delays $Ka = 0$, $Km = 2$.

This realizes the augmented IIR filter for L = 2, but with an additional 4 delays. The equation is

$$z^{-4}Y = z^{-4}\left[b_{11}z^{-2}Y + b_{12}z^{-3}Y + a_{11}X + a_{12}z^{-1}X + a_{13}z^{-2}X + a_{14}z^{-3}X\right]. \quad (4.78)$$

This realization is shown in Figure 34 on page 66.

b. $Ka = 1, Km = 0$

$$\begin{aligned} L &= 2Ka + Km \\ &= 2(1) + 0 \\ &= 2 \end{aligned}$$

For these values, we us^ pipeline modules with delays $Ka = 1$, $Km = 0$.

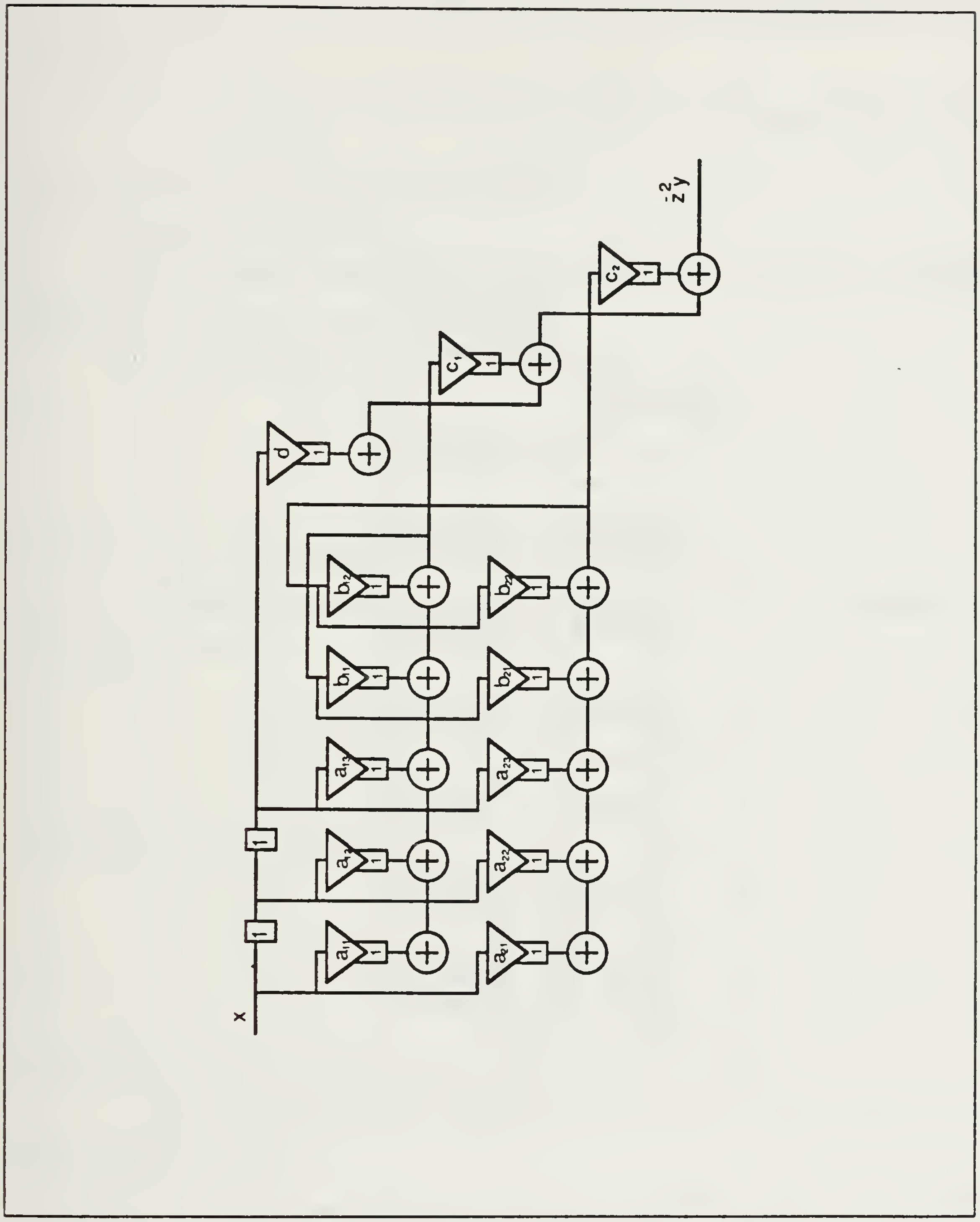

Figure 33. Second Order IIR Filter: L = 1 with Ka = 0, Km = 1

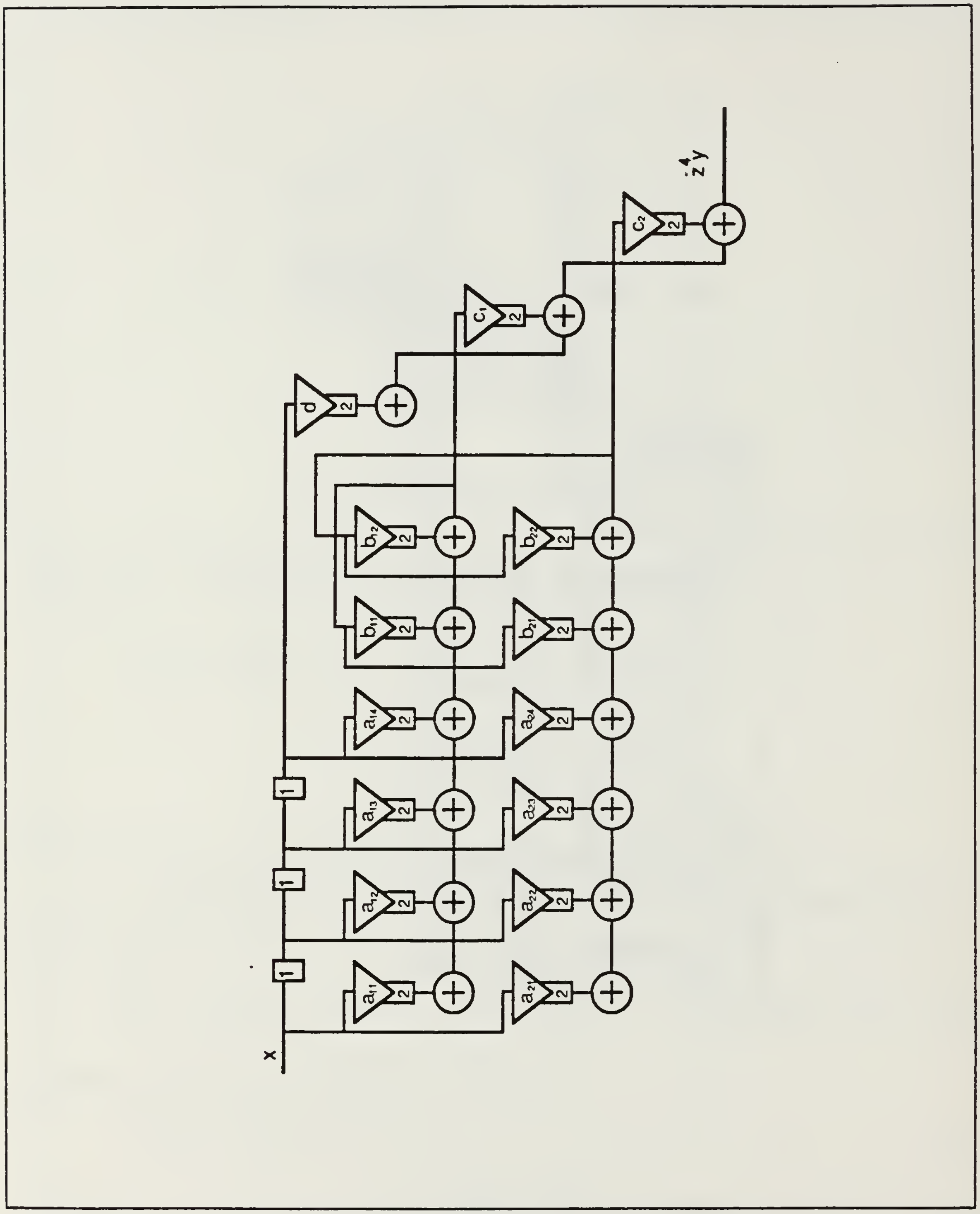

Figure 34. The Augmented IIR Filter: L = 2 with Ka = 0, Km = 2

This realizes the augmented IIR filter for L = 2, but with an additional 9 delays. The equation is

$$z^{-9}Y = z^{-9}\left[b_{11}z^{-2}Y + b_{12}z^{-3}Y + a_{11}X + a_{12}z^{-1}X + a_{13}z^{-2}X + a_{14}z^{-3}X\right]. \quad (4.79)$$

This realization is shown in Figure 35 on page 68.

7. Example for L = 3:

The only integer, non-negative values for Ka and Km which solve equation (4.76) are $Ka = 0, Km = 3$ or $Ka = 1, Km = 1$.

a. Ka = 0, Km = 3

$$\begin{aligned} L &= 2Ka + Km \\ &= 2(0) + 3 \\ &= 3 \end{aligned}$$

For these values, we use pipeline modules with delays $Ka = 0, Km = 3$.

This realizes the augmented IIR filter for L = 3, but with an additional 6 delays. The equation is

$$z^{-6}Y = z^{-6}\begin{bmatrix} b_{11}z^{-3}Y + b_{12}z^{-4}Y + a_{11}X + a_{12}z^{-1}X \\ + a_{13}z^{-2}X + a_{14}z^{-3}X + a_{15}z^{-4}X \end{bmatrix}. \quad (4.80)$$

This realization is shown in Figure 36 on page 69.

b. Ka = 1, Km = 1

$$\begin{aligned} L &= 2Ka + Km \\ &= 2(1) + 1 \\ &= 3 \end{aligned}$$

For these values, we use pipeline modules with delays $Ka = 1, Km = 1$.

This realizes the augmented IIR filter for L = 3, but with an additional 12 delays. The equation is

$$z^{-12}Y = z^{-12}\begin{bmatrix} b_{11}z^{-3}Y + b_{12}z^{-4}Y + a_{11}X + a_{12}z^{-1}X \\ + a_{13}z^{-2}X + a_{14}z^{-3}X + a_{15}z^{-4}X \end{bmatrix}. \quad (4.81)$$

This realization is shown in Figure 37 on page 70.

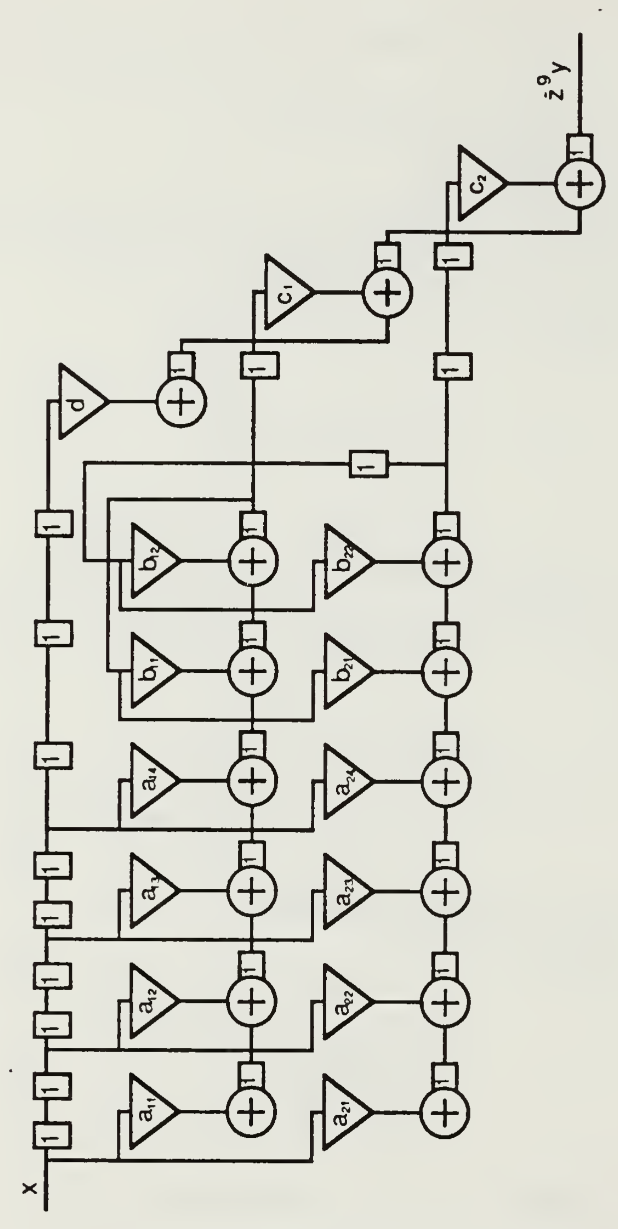

Figure 35. The Augmented IIR Filter: L = 2 with Ka = 1, Km = 0

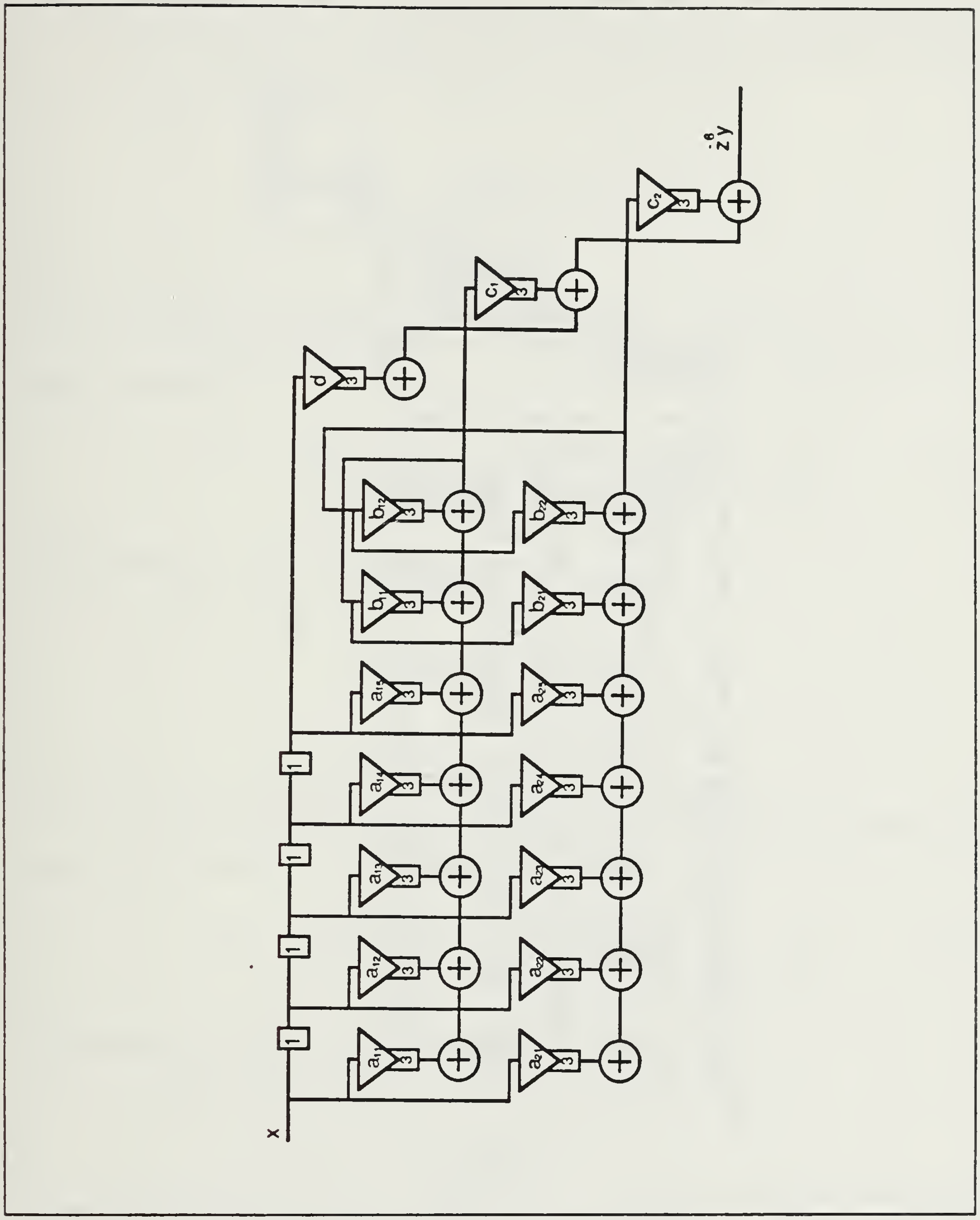

Figure 36. The Augmented IIR Filter: L = 3 with Ka = 0, Km = 3

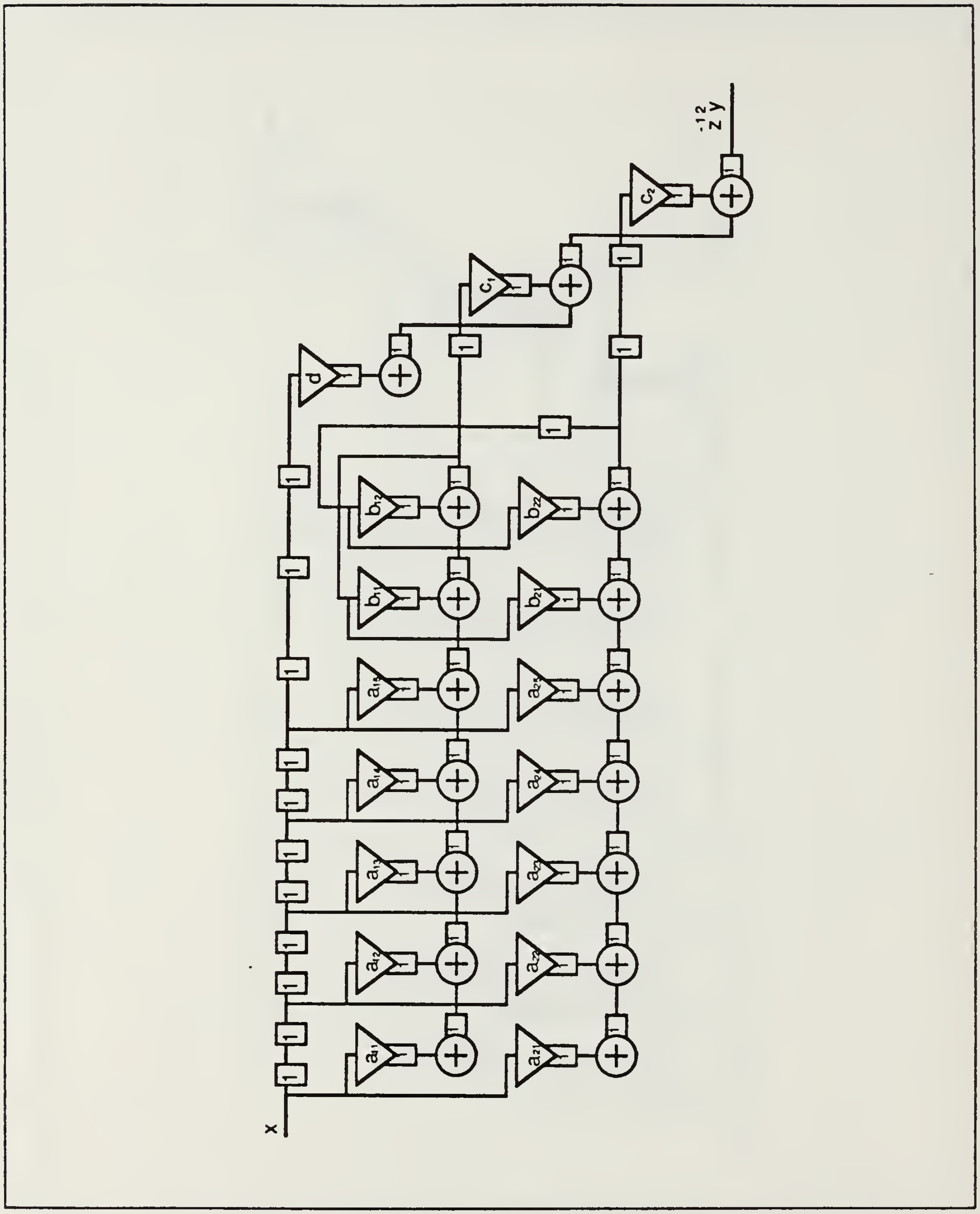

Figure 37. The Augmented IIR Filter: L = 3 with Ka = 1, Km = 1

8. **Example for L = 4:**

The only integer, non-negative values for Ka and Km which solve equation (4.76) are $Ka = 0$, $Km = 4$ and $Ka = 1$, $Km = 2$ or $Ka = 2$, $Km = 0$.

a. Ka = 0, Km = 4

$$\begin{aligned} L &= 2Ka + Km \\ &= 2(0) + 4 \\ &= 4 \end{aligned}$$

For these values, we use pipeline modules with delays $Ka = 0$, $Km = 4$.

This realizes the augmented IIR filter for L = 4, but with an additional 8 delays. The equation is

$$z^{-8}Y = z^{-8}\begin{bmatrix} b_{11}z^{-4}Y + b_{12}z^{-5}Y + a_{11}X + a_{12}z^{-1}X \\ + a_{13}z^{-2}X + a_{14}z^{-3}X + a_{15}z^{-4}X + a_{16}z^{-5}X \end{bmatrix}. \tag{4.82}$$

This realization is shown in Figure 38 on page 72.

b. Ka = 1, Km = 2

$$\begin{aligned} L &= 2Ka + Km \\ &= 2(1) + 2 \\ &= 4 \end{aligned}$$

For these values, we use pipeline modules with delays $Ka = 1$, $Km = 2$.

This realizes the augmented IIR filter for L = 4 but with an additional 15 delays. The equation is

$$z^{-15}Y = z^{-15}\begin{bmatrix} b_{11}z^{-4}Y + b_{12}z^{-5}Y + a_{11}X + a_{12}z^{-1}X \\ + a_{13}z^{-2}X + a_{14}z^{-3}X + a_{15}z^{-4}X + a_{16}z^{-5}X \end{bmatrix}. \tag{4.83}$$

This realization is shown in Figure 39 on page 73.

c. Ka = 2, Km = 0

$$\begin{aligned} L &= 2Ka + Km \\ &= 2(2) + 0 \\ &= 4 \end{aligned}$$

For these values, we use pipeline modules with delays $Ka = 2$, $Km = 0$.

This realizes the augmented IIR filter for L = 4, but with an additional 22 delays. The equation is

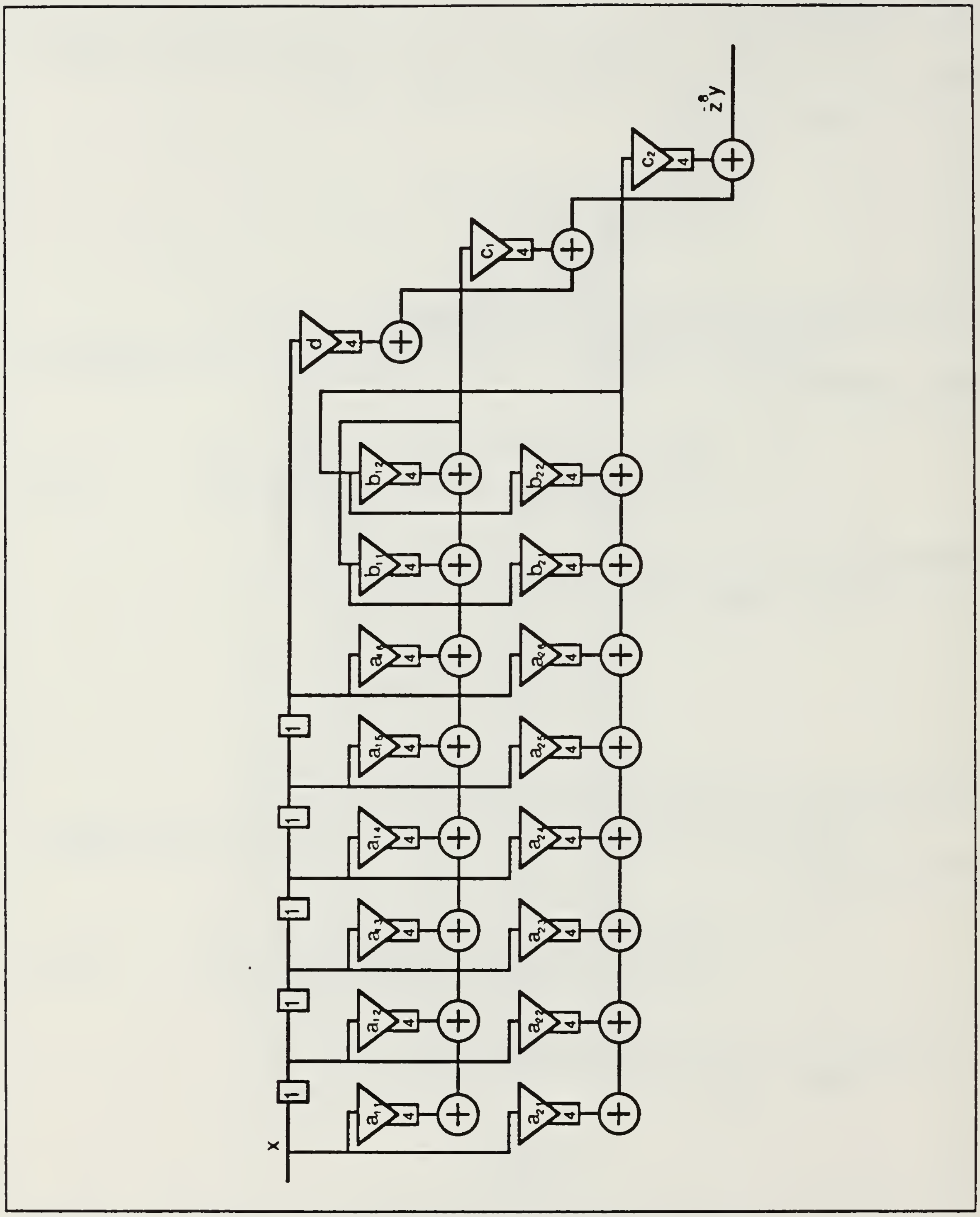

Figure 38. The Augmented IIR Filter: L = 4 with Ka = 0, Km = 4

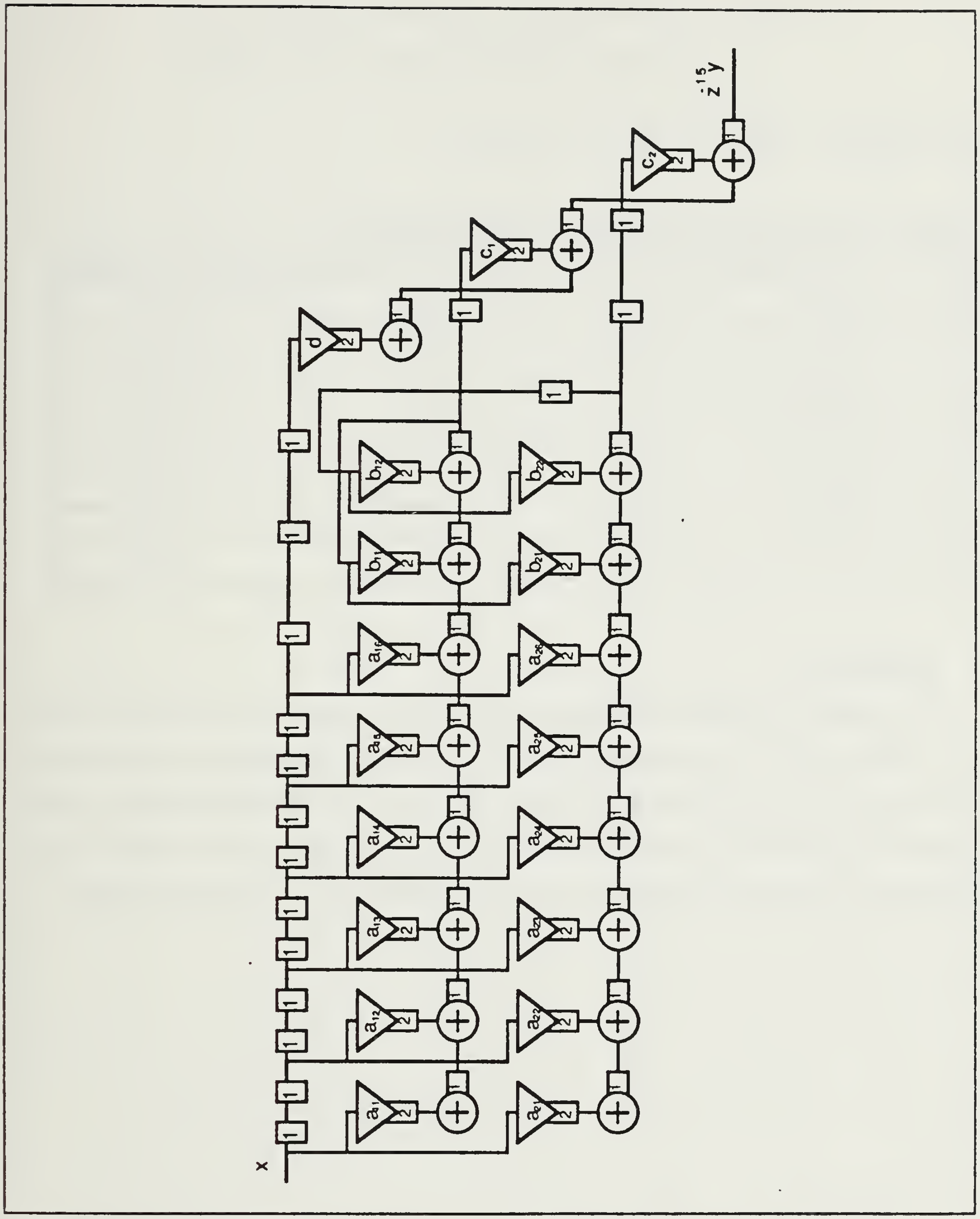

Figure 39. The Augmented IIR Filter: L = 4 with Ka = 1, Km = 2

$$z^{-22}Y = z^{-22}\begin{bmatrix} b_{11}z^{-4}Y + b_{12}z^{-5}Y + a_{11}X + a_{12}z^{-1}X \\ + a_{13}z^{-2}X + a_{14}z^{-3}X + a_{15}z^{-4}X + a_{16}z^{-5}X \end{bmatrix}. \tag{4.84}$$

This realization is shown in Figure 40 on page 75.

Table 5. STATE SPACE REALIZATION SUMMARY

L	Ka-add units	Km-multipliers	D(output delayed)	Figure
1	0	1	2	31
2	0	2	4	32
2	1	0	9	33
3	0	3	6	34
3	1	1	12	35
4	0	4	8	36
4	1	2	15	37
4	2	0	22	38

F. SUMMARY

This chapter develops an alternate method for high speed implementation of state space digital filters. The proposed method implements A^L instead of A as seen in equation (4.33). The proposed method allows arbitrary A in the implementation, and it can take full advantage of the optimum state space filters along with the benefit of A^L realization. The delay and complexity of each realization are summarized in Table 5.

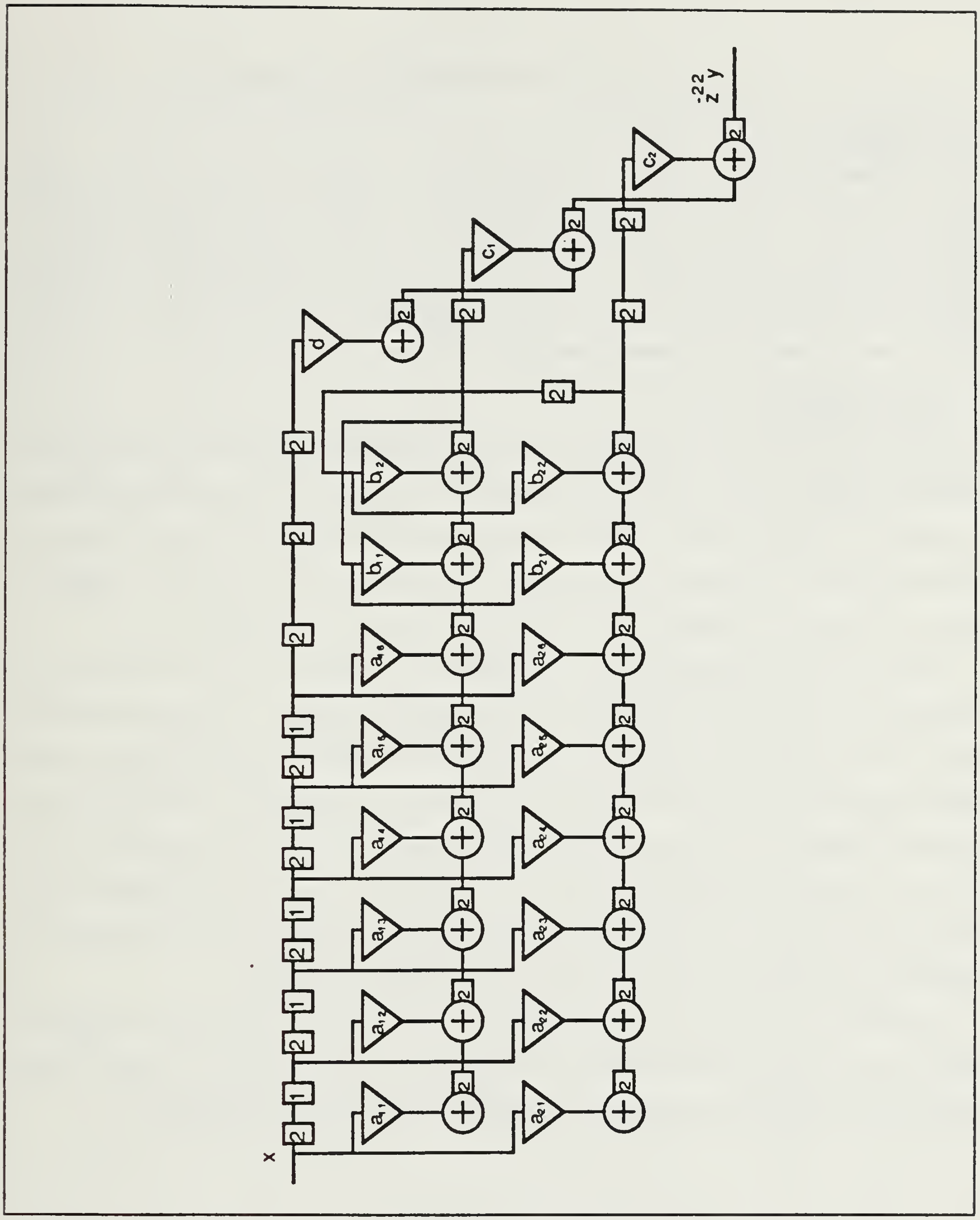

Figure 40. The Augmented IIR Filter: L = 4 with Ka = 2, Km = 0

V. CONCLUSIONS

A. SUMMARY

This thesis describes pipeline realizations of the IIR notch filter. The transfer function and the coefficients that are appropiate for IIR notch filter have been presented. Realization of a high speed IIR notch filter using a pipeline technique by Loomis and Sinha has been presented. An alternate technique in using state space realization by R. Gnanasekaran has also been presented.

B. COMPARISON

Table 4 in Chapter III summarizes the Loomis/Sinha pipeline recursive design and Table 5 in Chapter IV summarizes Gnanasekaran's state space realization. It was shown in Chapter III section I that we have only one choice of Ka and Km for each value of p and in Chapter IV section E that we have 1 choice of Ka and Km for L = 1, 2 choices of Ka and Km for L = 2 and for L = 3, and 3 choices of Ka and Km for L = 4.

In the Loomis/Sinha pipeline recursive design we note that for p = 2, Ka = 1, Km = 2, output delay = 5, the filter is unstable. However, in Gnanaskaran's state space realization for L = 4, Ka = 1, Km = 2, output delay = 15, the filter is stable. Thus we see that in cases where Loomis/Sinha realization are unstable, a state space realization always exists and is stable. However, it will require more hardware and have a greater overall delay. All of these results are summarized in Table 6. In that table, the column labelled D refers to the overall delay and the columns labelled #Mlt and #Add refer to the number of multipliers and number of adders as shown in the appropriate figure.

Note that for a given value of Ka and Km, the number of adders and multipliers required for the state space realization is larger than that required for the scalar. This disadvantage is offset by the guaranteed stability of the state space realization.

Table 6. SUMMARY OF REALIZATIONS

		State Space Realization					Scalar Realization				
Ka	Km	L	D	Fig	#Mlt	#Add	p	D	Fig	#Mlt	#Add
0	1	1	2	31	13	11	_	_	_	_	_
0	2	2	4	32	15	13	_	_	_	_	_
1	0	2	9	33	15	13	0	3	19	5	5
0	3	3	6	34	17	15	_	_	_	_	_
1	1	3	12	35	17	15	1	4	20	6	6
0	4	4	8	36	19	17	_	_	_	_	_
1	2	4	15	37	19	17	2	5	21	7	7
2	0	4	22	38	19	17	_	_	_	_	_
1	3	_	_	_	_	_	3	6	22	8	8

APPENDIX A. PROGRAM AND DATA OF FREQUENCY RESPONSE

A. PROGRAM FOR FREQUENCY RESPONSE CALCULATION

```
C                          DIGFREQ.FOR                 VERSION:  2/03/88
C
C
C  PURPOSE:  THIS PROGRAM COMPUTES THE FREQUENCY RESPONSE OF
C            DISCRETE SYSTEMS.  THE PROGRAM CONSISTS OF A MAIN
C            PROGRAM THAT CONTROLS THE INPUT/OUTPUT AND THE
C            SUBROUTINES dfresp AND coeff.  SUBROUTINE dfresp
C            COMPUTES THE FREQUENCY RESPONSE OF EACH SYSTEM.
C            SUBROUTINE coeff ALLOWS THE USER THE OPTION OF
C            GENERATING THE FILTER COEFFICIENTS OF THE SYSTEMS
C            TO BE ANALYZED BY WRITING THE APPROPRIATE EQUATIONS.
C            IF THE USER ELECTS TO GENERATE THE COEFFICIENTS BY
C            USING THE SUBROUTINE coeff, THE EQUATIONS MUST BE
C            WRITTEN INTO THE SUBROUTINE USING STANDARD FORTRAN 77
C            STATEMENTS.  THE COEFFICIENTS MUST BE STORED IN THE
C            ARRAYS b() AND c() WHICH CORRESPOND RESPECTIVELY TO THE
C            NUMERATOR AND DENOMINATOR TERMS OF THE SYSTEM EQUATION.
C            THE USER CAN SELECT ONE OF TWO OPERATING MODES:  BATCH
C            OR TEST.  IN BATCH MODE THE AMOUNT OF INTERFACE WITH
C            THE USER IS MINIMIZED AND IT IS ASSUMED THAT THE INPUT
C            DATA HAS BEEN STORED IN THE DEFAULT FILE 'DIGFREQ.IN'.
C            IN TEST MODE THE USER IS PROMPTED FOR THE NAME OF THE
C            INPUT FILE OR HAS THE OPTION TO PERFORM A TRIAL RUN BY
C            USING THE INPUT DATA STORED IN THE FILE 'DIGFREQ.TST'.
C            IT IS RECOMMENDED THAT FIRST-TIME USERS SELECT THE TEST
C            MODE AND MAKE A TRIAL RUN WITH THE PRESTORED INPUT DATA.
C            THE TEST MODE ECHOES THE INPUT DATA ONTO THE MONITOR TO
C            ALLOW VERIFICATION OF ITS ACCURACY.  THIS PROGRAM WILL
C            COMPUTE THE FREQUENCY RESPONSE OF UP TO THREE SYSTEMS.
C            FOR EACH SYSTEM, THE USER HAS THE OPTION OF HAVING THE
C            OUTPUT EXPRESSED IN DECIBELS (db).  THE OUTPUT OF THIS
C            PROGRAM IS STORED IN TABULAR FORM IN THE FILE
C            'DIGFREQ.OUT' AND IN A FORM SUITABLE FOR PLOTTING
C            IN THE FILE 'DIGFREQ.DAT'.
C
C
C*********************************** INPUT ************************************
C
C  THIS PROGRAM ASSUMES THAT EACH DISCRETE SYSTEM IS MODELED BY THE
C  EQUATION:  H(z) = num/den  WHERE:
C
C  num = b(0)*z**L + b(1)*z**(L-1) + ... + b(L-1)*z + b(L)
C
C  den = c(0)*z**N + c(1)*z**(N-1) + ... + c(N-1)*z + c(N)
C
C    L = A NON-NEGATIVE INTEGER, THE DEGREE OF THE NUMERATOR
C        POLYNOMIAL.
C    N = A NON-NEGATIVE INTEGER, THE DEGREE OF THE DENOMINATOR
```

```
C        POLYNOMIAL.
C    b(0)...b(L) = REAL COEFFICIENTS OF THE NUMERATOR TERMS.
C    c(0)...c(N) = REAL COEFFICIENTS OF THE DENOMINATOR TERMS.
C
C    THE INPUT PARAMETERS SHOULD BE STORED IN A FILE NAMED
C    'DIGFREQ.IN'.  ALL OF THE READ STATEMENTS USED BY THIS PROGRAM
C    REQUIRE FORMATTED INPUT.  PARTICULAR ATTENTION SHOULD BE PAID
C    TO THE FORMATS, ESPECIALLY THE USE OF THE DECIMAL POINT TO
C    DENOTE 'REAL' NUMBERS.  THE INPUT PARAMETERS REQUIRED BY THE
C    PROGRAM ARE LISTED BELOW.
C
C
C  NAME         TYPE       RANGE (ARRAYS)          RESTRICTIONS
C  ----         ----       --------------        ----------------
C  numsys      INTEGER                           1 <= numsys <= 3
C  L           INTEGER                             0 <= L <= 128
C  N           INTEGER                             0 <= N <= 128
C  dsorce      CHARACTER                            'F' OR 'S'
c  yscal       CHARACTER                          'STD' OR 'LOG'
C  theta0      REAL
C  dlthta      REAL
C  numpts      INTEGER                           1 <= numpts <= 101
C  b()         REAL        0, 1, 2, ..., L         0 <= L <= 128
C  c()         REAL        0, 1, 2, ..., N         0 <= N <= 128
C
C  WHERE:
C  numsys = THE NUMBER OF DISTINCT SYSTEMS H(z) TO BE ANALYZED.
C           THIS INTEGER VALUE MUST OCCUR AT THE TOP OF THE INPUT
C           FILE. IT DELINEATES THE NUMBER OF SYSTEMS TO BE READ BY
C           THE PROGRAM AND ANALYZED. FOR EACH SYSTEM (1, ..., numsys)
C           THE PARAMETERS BELOW MUST APPEAR IN THE INPUT FILE.
C
C  L = AN INTEGER VALUE SPECIFYING THE DEGREE OF THE NUMERATOR
C      POLYNOMIAL.
C
C  N = AN INTEGER VALUE SPECIFYING THE DEGREE OF THE DENOMINATOR
C      POLYNOMIAL.
C
C  dsorce = THE CHARACTER STRING 'F' OR 'S' DENOTING WHETHER THE
C           SYSTEM COEFFICIENTS ARE TO BE READ FROM THE INPUT FILE (F)
C           OR GENERATED (S) THROUGH USE OF THE SUBROUTINE coeff.
C
C  yscal = A CHARACTER STRING SPECIFYING THE DESIRED MAGNITUDE OPTION:
C          'STD' WILL PRODUCE STANDARD MAGNITUDE OUTPUT;
C          'LOG' WILL PRODUCE MAGNITUDE EXPRESSED IN DECIBELS (db).
C
C  theta0 = THE STARTING VALUE OF THETA (RADS) AS IN Z=EXP(J*THETA).
C
C  dlthta = THE INCREMENT OF THETA (RADIANS).
C
C  numpts = THE NUMBER OF FREQUENCY POINTS FOR WHICH THE OUTPUT IS
C           TO BE COMPUTED.
C
C  b() =  THE NUMERATOR COEFFICIENTS IN ORDER b(0), b(1), ..., b(L).
C         IF dsorce = 'F' IS SELECTED THEN THE USER MUST SUPPLY THE
C         L+1 NUMERATOR COEFFICIENTS IN THE FILE.  IF dsorce = 'S'
```

```
C         THEN THE USER HAS ELECTED TO GENERATE THE NUMERATOR
C         COEFFICIENTS BY WRITING THE APPROPRIATE FORTRAN STATEMENTS
C         IN THE SPACE PROVIDED IN SUBROUTINE coeff.  IF THIS METHOD
C         OF DATA GENERATION IS ELECTED THE PROGRAM MUST BE RECOMPILED
C         BEFORE EXECUTION.
C
C  c() =  THE DENOMINATOR COEFFICIENTS IN ORDER c(0), c(1), ..., c(N).
C         IF dsorce = 'F' IS SELECTED THEN THE USER MUST SUPPLY THE
C         N+1 DENOMINATOR COEFFICIENTS IN THE FILE.  IF dsorce = 'S'
C         THEN THE USER HAS ELECTED TO GENERATE THE DENOMINATOR
C         COEFFICIENTS BY WRITING THE APPROPRIATE FORTRAN STATEMENTS
C         IN THE SPACE PROVIDED IN SUBROUTINE coeff.  IF THIS METHOD
C         OF DATA GENERATION IS ELECTED THE PROGRAM MUST BE RECOMPILED
C         BEFORE EXECUTION.
C
C  NOTE:  THE INPUT FORMAT STATEMENTS OCCUR IN THE MAIN PROGRAM
C         FOLLOWING THE CAPTION: *********  INPUT FORMAT  *********.
C         THE FORM OF THE INPUT DATA FILE IS:
C
C  LINE #              ENTRIES                     FORMAT
C  ______              _______                     ______
C    1                 numsys                        i1
C    2              L,N,dsorce,yscal       i3,t11,i3,t21,a1,t31,a3
C    3            dlthta,theta0,numpts           2f10.0,i3
C  NOTE 1            b(k), k=0,1,...,L             6f10.0
C  NOTE 2            c(k), k=0,1,...,N             6f10.0
C  NOTE 3
C
C  WHERE:  NN = 1 + (L/6 ROUNDED DOWN TO THE NEXT SMALLER INTEGER).
C          ND = 1 + (N/6 ROUNDED DOWN TO THE NEXT SMALLER INTEGER).
C
C  NOTES 1.  THE NEXT NN LINES ARE ONLY REQUIRED IF dsorce = 'F'.  IF
C            dsorce = 'S' THEN THE USER HAS ELECTED TO GENERATE THE
C            L+1 NUMERATOR COEFFICIENTS IN THE SUBROUTINE coeff.
C            THE USER MUST PROVIDE THE APPROPRIATE FORTRAN STATEMENTS
C            IN SUBROUTINE coeff TO GENERATE THE VALUES FOR b().
C
C        2.  THE NEXT ND LINES ARE ONLY REQUIRED IF dsorce = 'F'.  IF
C            dsorce = 'S' THEN THE USER HAS ELECTED TO GENERATE THE
C            N+1 DENOMINATOR COEFFICIENTS IN THE SUBROUTINE coeff.
C            THE USER MUST PROVIDE THE APPROPRIATE FORTRAN STATEMENTS
C            IN SUBROUTINE coeff TO GENERATE THE VALUES FOR c().
C
C        3.  FOR numsys > 1 THE FORMAT OF LINES 2... IS REPEATED.
C
C        4.  THE FORMAT f10.0 USED FOR INPUT DATA PERMITS THE DECIMAL
C            POINT TO BE PLACED ANYWHERE IN THE FIELD OF 10 COLUMNS
C            AND ALSO ALLOWS THE EXPONENTIAL FORMAT TO BE USED (EG.
C            3146.2 = 3.1462E+03).
C
C
C*******************************  OUTPUT  *******************************
C
C  THE OUTPUT DATA CREATED BY THE PROGRAM IS STORED IN TABULAR FORM
C  IN THE FILE 'DIGFREQ.OUT'.  ADDITIONALLY, THE OUTPUT DATA IS
C  WRITTEN INTO THE FILE 'DIGFREQ.DAT' TO FACILITATE PLOTTING BY
```

```
C  A SEPARATE, USER SUPPLIED PROGRAM.  THE FORMAT OF THE DATA IN
C  'DIGFREQ.DAT' IS:  e12.6, 2x, e12.6.  THE FIRST ENTRY CORRESPONDS
C  TO THE ORDINATE VALUE (THETA) AND THE SECOND ENTRY THE ABSCISSA
C  VALUE (MAGNITUDE OR PHASE).  ADDITIONAL HEADER INFORMATION IS
C  WRITTEN INTO THE DATA FILE TO ALLOW FOR CONTROL AND LABELING OF
C  EACH PLOT.
C
C
C********************************  EXAMPLE  ********************************
C
C  THE INPUT PARAMETERS FOR THE SYSTEM DESCRIBED BELOW ARE STORED IN
C  THE SAMPLE INPUT FILE 'DIGFREQ.TST' AND CAN BE USED FOR A TRIAL
C  RUN IN THE TEST MODE.
C
C
C  SYSTEM:  H(z)=z/(z-0.5)
C
C  GOAL:    TO OBTAIN THE FREQUENCY RESPONSE FOR THIS SYSTEM FROM
C           THETA = 0.0 TO THETA = 3.14159 (PI RADIANS) IN STEPS
C           OF dlthta = PI/10.0
C
C  FOR THE SYSTEM DESCRIBED ABOVE THE INPUT FILE IS:
C
C  1
C  001       001       F         STD
C  0.314159  0.0       011
C  1.0       0.0
C  1.0       -0.5
C
C
C  THE RESULTING OUTPUT DATA FILE: 'DIGFREQ.OUT' IS:
C
C                INPUT DATA FOR SYSTEM #  1
C
C  INPUT DATA SOURCEFILE: DIGFREQ.TST
C  DEGREE OF NUMERATOR =   1
C  DEGREE OF DENOMINATOR =   1
C  dsorce = F
C  NUMBER OF FREQUENCY POINTS =  11     MAGNITUDE OPTION = STD
C  STARTING VALUE OF THETA =  .000000E+00
C  INCREMENT OF THETA =  .314159E+00
C
C  THE NUMERATOR COEFFICIENTS b(0),b(1)...b(L) ARE
C
C   .1000E+01    .0000E+00
C
C
C  THE DENOMINATOR COEFFICIENTS c(0),c(1)...c(N) ARE
C
C   .1000E+01   -.5000E+00
C
C
C                  OUTPUT DATA FOR SYSTEM #  1
C
C         THETA        MAGNITUDE         PHASE
C       (RADIANS)                      (DEGREES)
```

```
C
C      .000000E+00     .200000E+01      .000000E+00
C      .314159E+00     .182897E+01     -.164149E+02
C      .628318E+00     .150588E+01     -.262677E+02
C      .942477E+00     .122886E+01     -.29807E+02
C      .125664E+01     .103088E+01     -.293546E+02
C      .157080E+01     .894428E+00     -.265651E+02
C      .188495E+01     .800894E+00     -.223862E+02
C      .219911E+01     .737654E+00     -.173608E+02
C      .251327E+01     .696900E+00     -.118186E+02
C      .282743E+01     .674038E+00     -.597793E+01
C      .314159E+01     .666667E+00     -.484184E-04
C
C  ------------   END OF RUN, SYSTEM # 1  ------------
C
C
C  FOR ILLUSTRATIVE PURPOSES THE COEFFICIENTS b() AND c() COULD
C  HAVE BEEN GENERATED BY SPECIFYING dsorce = 'S' AND WRITING THE
C  APPROPRIATE FORTRAN STATEMENTS INTO SUBROUTINE coeff.  THE
C  STATEMENTS THAT COULD BE USED TO ACCOMPLISH THIS ARE WRITTEN INTO
C  THE SUBROUTINE BUT ARE 'COMMENTED OUT'.
C
C****************************  MAIN PROGRAM  ****************************


      character infile*12, mode*1, ylabl*13, dsorce*1, yscal*3
      real mh(101), ph(101), thetav(101), c(0:128), b(0:128)

C  PROMPT USER FOR MODE: BATCH OR TEST.

      write(*,1115)
      read(*,1117) mode
      if((mode.eq.'Y').or.(mode.eq.'y')) then
mode = 'Y'
write(*,1118)
read(*,1119) infile
      else
infile = 'DIGFREQ.IN'
      endif

C  UNIT=1 DEFINED AS INPUT FILE.  UNITS=2,3 DEFINED AS OUTPUT FILES.

      open(unit=1,file=infile,status='old',iostat=ierr,err=999)
      open(unit=2,file='DIGFREQ.OUT')
      open(unit=3,file='DIGFREQ.DAT')

C  READ INPUT PARAMETERS AND CONDUCT ERROR CHECKS.

      read(1,1000) numsys
      numplts = numsys*2
      write(3,2000) numplts
      if((numsys.lt.1).or.(numsys.gt.3)) then
write(*,1122) numsys
stop 'Error, numsys must be in the range: 1 <= numsys <= 3.'
      endif
```

```
      do 10 nsys=1, numsys
data mh/101*0.0/, ph/101*0.0/, thetav/101*0.0/
data b/129*0.0/, c/129*0.0/

read(1,1001) L, N, dsorce, yscal
read(1,1002) dlthta, theta0, numpts

if((L.lt.0).or.(L.gt.128)) then
  write(*,1124) nsys, L
  stop 'Error, L must be in the range: 0 <= L <= 128.'
elseif((N.lt.0).or.(N.gt.128)) then
  write(*,1125) nsys, N
  stop 'Error, N must be in the range: 0 <= N <= 128.'
   endif

if((dsorce.eq.'F').or.(dsorce.eq.'f')) then
  dsorce = 'F'
elseif((dsorce.eq.'S').or.(dsorce.eq.'s')) then
  dsorce = 'S'
    else
  write(*,1018) dsorce
  stop 'The allowed values for dsorce are: ''S'' or ''F''.'
   endif

if((numpts.lt.1).or.(numpts.gt.101)) then
  write(*,1127) nsys, numpts
  stop 'Error, numpts must be in the range: 1 <= numpts <= 101.'
   endif

if((yscal.eq.'STD').or.(yscal.eq.'std')) then
  yscal = 'STD'
  ylabl = '  MAGNITUDE  '
elseif((yscal.eq.'LOG').or.(yscal.eq.'log')) then
  yscal = 'LOG'
  ylabl = 'MAGNITUDE(db)'
    else
  write(*,1128) yscal
  stop 'Error, yscal must be the string: ''LOG'' or ''STD''.'
   endif

C  FOR dsorce = 'F' READ THE COEFFICIENTS b() AND c() FROM THE INPUT
C  FILE.  FOR dsorce = 'S' CALL coeff TO GENERATE THE COEFFICIENTS.

if(dsorce.eq.'F') then
  read(1,1003) (b(k),k=0,L)
  read(1,1003) (c(k),k=0,N)
    else
  call coeff(L,N,nsys,b,c)
   endif

C  WRITE INPUT DATA INTO THE OUTPUT FILE: DIGFREQ.OUT.

write(2,1008) nsys
write(2,1010) infile
write(2,1110) L
write(2,1111) N
```

```
write(2,1019) dsorce
write(2,1112) numpts, yscal
write(2,1113) theta0
write(2,1114) dlthta
write(2,1004)
write(2,1005) (b(k),k=0,L)
write(2,1006)
write(2,1005) (c(k),k=0,N)
write(2,1009) nsys
write(2,1126) ylabl
write(2,1007)

C  FOR TEST MODE ECHO ALL INPUTS ONTO MONITOR (UNIT = *).

if(mode.eq.'Y') then
  write(*,1120) nsys, infile
  write(*,1110) L
  write(*,1111) N
  write(*,1019) dsorce
  write(*,1112) numpts, yscal
  write(*,1113) theta0
  write(*,1114) dlthta
  write(*,1004)
  write(*,1005) (b(k),k=0,L)
  write(*,1006)
  write(*,1005) (c(k),k=0,N)
  write(*,1123) nsys
  pause 'END OF RUN. STRIKE <CR> WHEN READY TO CONTINUE.'
   endif

C  CALL dfresp TO COMPUTE THE FREQUENCY RESPONSE.

call dfresp(b,c,mh,ph,L,N,theta0,dlthta,thetav,numpts,yscal)

C  WRITE RESULTS INTO OUTPUT FILE: DIGFREQ.DAT.

write(3,2001) numpts
write(3,*) 'MAGNITUDE RESPONSE'
write(3,*) 'THETA (RADS)'
write(3,2003) ylabl
do 55 np=1, numpts
  write(3,2010) thetav(np), mh(np)
55      continue

write(3,2001) numpts
write(3,*) 'PHASE RESPONSE'
write(3,*) 'THETA (RADS)'
write(3,2003) ' PHASE (DEG) '
do 56 np=1, numpts
  write(3,2010) thetav(np), ph(np)
56      continue

C  WRITE RESULTS INTO OUTPUT FILE: DIGFREQ.OUT.

do 150 np=1, numpts
```

```
  write(2,1013) thetav(np), mh(np), ph(np)
150     continue
write(2,1123) nsys

10    continue

      write(*,1121)
999   close(unit=1)
      close(unit=2)
      close(unit=3)

      if(ierr.gt.0) then
write(*,1116) ierr
      endif

C**********  INPUT FORMAT  ***************

1000  format(i1)
1001  format(i3,t11,i3,t21,a1,t31,a3)
1002  format(2f10.0,i3)
1003  format(6f10.0)

C*******************************************

1004  format(t4,'THE NUMERATOR COEFFICIENTS b(0),b(1)...b(L) ARE: ',/)
1005  format(6(2X,e11.4),//)
1006 0format(//,t4,'THE DENOMINATOR COEFFICIENTS c(0),c(1)...c(N)',
     1' ARE: ',/)
1007  format(t6,'(RADIANS)',t38,'(DEGREES)',/)
1008  format(t16,' INPUT DATA FOR SYSTEM # ',i1,//)
1009  format(///,t16,' OUTPUT DATA FOR SYSTEM # ',i1,/)
1010  format(t4,'INPUT DATA SOURCEFILE: ',a12)
1013  format(t4,3(e12.6,4x))
1018  format(1x,'dsorce = ',a1,2x,'Error, illegal value for dsorce.')
1019  format(t4,'dsorce = ',a1)
1110  format(t4,'DEGREE OF NUMERATOR = ',i3)
1111  format(t4,'DEGREE OF DENOMINATOR = ',i3)
1112 0format(t4,'NUMBER OF FREQUENCY POINTS = ',i3,t39,'MAGNITUDE',
     1' OPTION = ',a3)
1113  format(t4,'STARTING VALUE OF THETA = ',e12.6)
1114  format(t4,'INCREMENT OF THETA = ',e12.6,/)
1115 0format(1x,'DO YOU WISH TO RUN THIS PROGRAM IN TEST',
     1' MODE ?    (Y/N) <CR> : ', ,)
1116 0format(///,1x,'ERROR OPENING INPUT FILE, PROGRAM TERMINATED.',
     1//,1x,'ERROR CODE:',i4,/////)
1117  format(a1)
1118 0format(/////,1x,'TYPE THE NAME OF YOUR DATA FILE FOLLOWED',
     1' BY <CR>.',/,' IF YOU DESIRE TO MAKE A TEST RUN USING THE',
     2' SAMPLE DATA ALREADY STORED',/,' IN THE FILE: DIGFREQ.TST',
     3 '        TYPE: DIGFREQ.TST <CR>',/,' FILENAME: ', ,)
1119  format(a12)
1120  format(////,t4,'SYSTEM # ',i1,'    INPUT DATA SOURCEFILE: ',a12)
1121 0format(//,t4,'TABULAR OUTPUT DATA IS STORED IN FILE: DIGFREQ.OUT',
     1/,t4,'PLOTTING DATA IS STORED IN FILE: DIGFREQ.DAT. ')
1122  format(//////,t2,'The value of numsys is: ',i1,'.')
```

```
1123  format(/,1x,13('-'),'  END OF RUN, SYSTEM # ',i1,2x,13('-'),//)
1124 0format(//////,t2,'The degree(L) of the numerator for system ',
     1'# ',i1,' is :  L = ',i3,'.')
1125 0format(//////,t2,'The degree(N) of the denominator for system'
     1,' # ',i1,' is  : N = ',i3,'.')
1126  format(///,t8,'THETA',t21,a13,t40,'PHASE')
1127  format(//////,t2,'The value of numpts for system ',i1,' is: ',i3)
1128  format(/////,t2,'The value of yscal is: ',a3,'.')
2000  format(i1)
2001  format(i3)
2003  format(a13)
2010  format(e12.6,2x,e12.6)

      end


C                        SUBROUTINE: dfresp


C  PURPOSE:  THIS SUBROUTINE COMPUTES THE FREQUENCY RESPONSE OF
C            THE SYSTEM.  ALL FREQUENCY CALCULATIONS ARE IN RADIANS,
C            HOWEVER THE OUTPUT IS CONVERTED TO DEGREES.
C            THE OUTPUT FORMAT FOR EACH FREQUENCY INCREMENT IS:
C            MAGNITUDE(M)    PHASE(P)      AS IN:  M*EXP(J*P).


      subroutine dfresp(b,c,mh,ph,L,N,theta0,dlthta,thetav,numpts,yscal)
      real mh(numpts), ph(numpts), thetav(numpts), imz, rez
      real b(0:L), c(0:N)
      character yscal*3
      complex z, den, num, h, ci

C  DEFINE CONSTANTS.

      pi = 4.0*atan(1.0)
      ci = (1.0,0.0)

C  ITERATE FROM theta0, IN INCREMENTS OF dlthta.

      do 100 np=1, numpts
num = ci*b(0)
den = ci*c(0)
thetav(np) = theta0 + (np-1)*dlthta
rez = cos(thetav(np))
imz = sin(thetav(np))
z = cmplx(rez,imz)

C  CALCULATE NUMERATOR FOR GIVEN VALUE OF THETA, IF L > 0.

if(L.gt.0) then
  do 50 k=1, L
    num = z*num + ci*b(k)
 50       continue
   endif
```

```
C  CALCULATE DENOMINATOR FOR GIVEN VALUE OF THETA, IF N > 0.

if(N.gt.0) then
  do 70 k=1, N
    den = z*den + ci*c(k)
 70       continue
   endif
h = num/den

C  CONVERT COMPLEX VALUE 'h' INTO MAGNITUDE(mh) AND PHASE(ph) TERMS.
C  IF yscal = 'LOG' THEN CONVERT MAGNITUDE TO DECIBELS (db).
C  DIVIDE BY ZERO AVOIDED BY 'if' STATEMENTS.

mh(np) = cabs(h)
if(yscal.eq.'LOG') then
  if(mh(np).gt.0.00001) then
    mh(np) = 20.0*log10(mh(np))
     else
    mh(np) = -100.0
   endif
   endif

if(abs(real(h)).lt.1.0e-15) then
  if(abs(aimag(h)).le.1.0e-15) ph(np)=0.0
  if(aimag(h).gt.1.0e-15) ph(np)=90.0
  if(aimag(h).lt.-1.0e-15) ph(np)=-90.0
     else
  ph(np) = (180.0/pi)*atan2(aimag(h),real(h))
   endif

 100  continue

      return
      end



C                         SUBROUTINE: coeff


C  PURPOSE:  THIS SUBROUTINE ALLOWS THE USER TO GENERATE THE
C            NUMERATOR AND DENOMINATOR COEFFICIENTS THAT DESCRIBE
C            EACH SYSTEM TO BE ANALYZED.  IF dsorce = 'S' THEN
C            THE MAIN PROGRAM WILL CALL THIS SUBROUTINE.


      subroutine coeff(L,N,nsys,b,c)
      real b(0:L), c(0:N)

      pi = 4.0*atan(1.0)
      el = L
      en = N

C*************************************************************************
C  DEVELOP THE EQUATIONS TO GENERATE VALUES FOR THE ARRAYS b() AND c()
C  IN THIS SPACE.  THE STATEMENTS TYPED IN MUST FOLLOW STANDARD
```

```
C  FORTRAN 77 RULES AND MAY USE FORTRAN 77 INTRINSIC FUNCTIONS SUCH AS:
C  SIN(), COS(), ABS()...  AN EXAMPLE IS SHOWN BELOW.  NOTE THAT THE
C  VALUE nsys CAN BE USED TO DISTINGUISH BETWEEN SYSTEMS IF MORE THAN
C  ONE SYSTEM (numsys > 1) IS TO BE ANALYZED.
C
C ***  EXAMPLE  ***
C
C      if(nsys.eq.1) then
C        do 2 i=0, L
C          b(i) = cos(i*pi/(2.0*el))
C 2      continue
C        do 3 i=0, N
C          c(i) = cos(2.0*i*pi/(3.0*en))
C 3      continue
C      endif




C*******************************************************************************

      return
      end
```

B. MAGNITUDE, PHASE OF SECOND ORDER IIR NOTCH FILTER: P=0

```
                INPUT DATA FOR SYSTEM # 1


   INPUT DATA SOURCEFILE:  digfreq.tst
   DEGREE OF NUMERATOR =    2
   DEGREE OF DENOMINATOR =    2
   dsorce = F
   NUMBER OF FREQUENCY POINTS = 100   MAGNITUDE OPTION = STD
   STARTING VALUE OF THETA =  .000000E+00
   INCREMENT OF THETA =  .280600E-01

   THE NUMERATOR COEFFICIENTS b(0),b(1)...b(L) ARE:

    .9900E+00   -.3299E+00    .9900E+00


   THE DENOMINATOR COEFFICIENTS c(0),c(1)...c(N) ARE:

    .1000E+01   -.3299E+00    .9801E+00
```

OUTPUT DATA FOR SYSTEM # 1

THETA (RADIANS)	MAGNITUDE	PHASE (DEGREES)
.000000E+00	.999939E+00	.000000E+00
.280600E-01	.999939E+00	-.193947E-01
.561200E-01	.999939E+00	-.388289E-01
.841800E-01	.999939E+00	-.583436E-01
.112240E+00	.999938E+00	-.779784E-01
.140300E+00	.999938E+00	-.977754E-01
.168360E+00	.999937E+00	-.117779E+00
.196420E+00	.999936E+00	-.138030E+00
.224480E+00	.999935E+00	-.158581E+00
.252540E+00	.999934E+00	-.179474E+00
.280600E+00	.999933E+00	-.200769E+00
.308660E+00	.999931E+00	-.222516E+00
.336720E+00	.999930E+00	-.244780E+00
.364780E+00	.999928E+00	-.267620E+00
.392840E+00	.999926E+00	-.291112E+00
.420900E+00	.999923E+00	-.315332E+00
.448960E+00	.999920E+00	-.340366E+00
.477020E+00	.999917E+00	-.366300E+00
.505080E+00	.999914E+00	-.393251E+00
.533140E+00	.999910E+00	-.421327E+00
.561200E+00	.999906E+00	-.450663E+00
.589260E+00	.999902E+00	-.481405E+00
.617320E+00	.999896E+00	-.513723E+00
.645380E+00	.999891E+00	-.547804E+00
.673440E+00	.999884E+00	-.583872E+00
.701500E+00	.999876E+00	-.622182E+00
.729560E+00	.999868E+00	-.663033E+00
.757620E+00	.999858E+00	-.706764E+00
.785680E+00	.999847E+00	-.753788E+00
.813740E+00	.999835E+00	-.804587E+00
.841800E+00	.999820E+00	-.859743E+00
.869860E+00	.999803E+00	-.919964E+00
.897920E+00	.999783E+00	-.986100E+00
.925980E+00	.999759E+00	-.105921E+01
.954040E+00	.999731E+00	-.114061E+01
.982100E+00	.999697E+00	-.123199E+01
.101016E+01	.999655E+00	-.133548E+01
.103822E+01	.999603E+00	-.145389E+01
.106628E+01	.999537E+00	-.159095E+01
.109434E+01	.999453E+00	-.175176E+01
.112240E+01	.999343E+00	-.194344E+01
.115046E+01	.999193E+00	-.217626E+01
.117852E+01	.998985E+00	-.246560E+01
.120658E+01	.998681E+00	-.283559E+01
.123464E+01	.998213E+00	-.332633E+01
.126270E+01	.997440E+00	-.400969E+01
.129076E+01	.996024E+00	-.502855E+01

.131882E+01	.992994E+00	-.671241E+01
.134688E+01	.984530E+00	-.100259E+02
.137494E+01	.942857E+00	-.194059E+02
.140300E+01	.381903E-01	-.877665E+02
.143106E+01	.939926E+00	.200014E+02
.145912E+01	.984111E+00	.102595E+02
.148718E+01	.992867E+00	.687161E+01
.151524E+01	.995971E+00	.516115E+01
.154330E+01	.997413E+00	.413001E+01
.157136E+01	.998198E+00	.344006E+01
.159942E+01	.998672E+00	.294546E+01
.162748E+01	.998979E+00	.257310E+01
.165554E+01	.999190E+00	.228226E+01
.168360E+01	.999341E+00	.204852E+01
.171166E+01	.999452E+00	.185629E+01
.173972E+01	.999537E+00	.169521E+01
.176778E+01	.999603E+00	.155808E+01
.179584E+01	.999655E+00	.143978E+01
.182390E+01	.999698E+00	.133653E+01
.185196E+01	.999732E+00	.124550E+01
.188002E+01	.999761E+00	.116455E+01
.190808E+01	.999785E+00	.109198E+01
.193614E+01	.999805E+00	.102647E+01
.196420E+01	.999822E+00	.966960E+00
.199226E+01	.999837E+00	.912579E+00
.202032E+01	.999850E+00	.862632E+00
.204838E+01	.999861E+00	.816531E+00
.207644E+01	.999871E+00	.773792E+00
.210450E+01	.999880E+00	.734015E+00
.213256E+01	.999888E+00	.696843E+00
.216062E+01	.999894E+00	.661987E+00
.218868E+01	.999901E+00	.629193E+00
.221674E+01	.999906E+00	.598246E+00
.224480E+01	.999911E+00	.568950E+00
.227286E+01	.999915E+00	.541144E+00
.230092E+01	.999919E+00	.514683E+00
.232898E+01	.999923E+00	.489434E+00
.235704E+01	.999926E+00	.465285E+00
.238510E+01	.999929E+00	.442146E+00
.241316E+01	.999932E+00	.419918E+00
.244122E+01	.999934E+00	.398521E+00
.246928E+01	.999937E+00	.377889E+00
.249734E+01	.999939E+00	.357941E+00
.252540E+01	.999941E+00	.338639E+00
.255346E+01	.999942E+00	.319915E+00
.258152E+01	.999944E+00	.301724E+00
.260958E+01	.999945E+00	.284024E+00
.263764E+01	.999947E+00	.266771E+00
.266570E+01	.999948E+00	.249929E+00
.269376E+01	.999949E+00	.233459E+00
.272182E+01	.999950E+00	.217332E+00
.274988E+01	.999951E+00	.201519E+00
.277794E+01	.999952E+00	.185987E+00

------------ END OF RUN, SYSTEM # 1 ------------

C. MAGNITUDE, PHASE OF THE AUGMENTED IIR NOTCH FILTER: P = 1

INPUT DATA FOR SYSTEM # 1

INPUT DATA SOURCEFILE: digfreq.tst
DEGREE OF NUMERATOR = 3
DEGREE OF DENOMINATOR = 3
dsorce = F
NUMBER OF FREQUENCY POINTS = 100 MAGNITUDE OPTION = STD
STARTING VALUE OF THETA = .000000E+00
INCREMENT OF THETA = .280600E-01

THE NUMERATOR COEFFICIENTS b(0),b(1)...b(L) ARE:

.9900E+00 -.3299E-02 .8811E+00 .3266E+00

THE DENOMINATOR COEFFICIENTS c(0),c(1)...c(N) ARE:

.1000E+01 .0000E+00 .8712E+00 .3234E+00

OUTPUT DATA FOR SYSTEM # 1

THETA (RADIANS)	MAGNITUDE	PHASE (DEGREES)
.000000E+00	.999940E+00	.000000E+00
.280600E-01	.999940E+00	-.193950E-01
.561200E-01	.999939E+00	-.388293E-01
.841800E-01	.999939E+00	-.583432E-01
.112240E+00	.999939E+00	-.779798E-01
.140300E+00	.999938E+00	-.977766E-01
.168360E+00	.999937E+00	-.117780E+00
.196420E+00	.999936E+00	-.138031E+00
.224480E+00	.999935E+00	-.158579E+00
.252540E+00	.999934E+00	-.179475E+00
.280600E+00	.999933E+00	-.200768E+00
.308660E+00	.999932E+00	-.222515E+00
.336720E+00	.999930E+00	-.244780E+00
.364780E+00	.999928E+00	-.267618E+00
.392840E+00	.999926E+00	-.291112E+00
.420900E+00	.999923E+00	-.315332E+00
.448960E+00	.999921E+00	-.340364E+00
.477020E+00	.999918E+00	-.366303E+00

.505080E+00	.999914E+00	-.393250E+00
.533140E+00	.999911E+00	-.421327E+00
.561200E+00	.999906E+00	-.450665E+00
.589260E+00	.999902E+00	-.481406E+00
.617320E+00	.999897E+00	-.513721E+00
.645380E+00	.999891E+00	-.547805E+00
.673440E+00	.999884E+00	-.583875E+00
.701500E+00	.999877E+00	-.622183E+00
.729560E+00	.999868E+00	-.663032E+00
.757620E+00	.999859E+00	-.706764E+00
.785680E+00	.999848E+00	-.753788E+00
.813740E+00	.999835E+00	-.804590E+00
.841800E+00	.999820E+00	-.859746E+00
.869860E+00	.999803E+00	-.919964E+00
.897920E+00	.999783E+00	-.986101E+00
.925980E+00	.999759E+00	-.105921E+01
.954040E+00	.999731E+00	-.114061E+01
.982100E+00	.999697E+00	-.123199E+01
.101016E+01	.999655E+00	-.133548E+01
.103822E+01	.999603E+00	-.145389E+01
.106628E+01	.999538E+00	-.159095E+01
.109434E+01	.999454E+00	-.175176E+01
.112240E+01	.999343E+00	-.194344E+01
.115046E+01	.999194E+00	-.217625E+01
.117852E+01	.998985E+00	-.246560E+01
.120658E+01	.998681E+00	-.283559E+01
.123464E+01	.998214E+00	-.332632E+01
.126270E+01	.997441E+00	-.400968E+01
.129076E+01	.996025E+00	-.502853E+01
.131882E+01	.992995E+00	-.671239E+01
.134688E+01	.984532E+00	-.100259E+02
.137494E+01	.942860E+00	-.194058E+02
.140300E+01	.382032E-01	-.877594E+02
.143106E+01	.939922E+00	.200013E+02
.145912E+01	.984110E+00	.102595E+02
.148718E+01	.992866E+00	.687158E+01
.151524E+01	.995970E+00	.516112E+01
.154330E+01	.997413E+00	.412999E+01
.157136E+01	.998197E+00	.344004E+01
.159942E+01	.998671E+00	.294544E+01
.162748E+01	.998979E+00	.257309E+01
.165554E+01	.999189E+00	.228225E+01
.168360E+01	.999340E+00	.204850E+01
.171166E+01	.999452E+00	.185627E+01
.173972E+01	.999537E+00	.169520E+01
.176778E+01	.999603E+00	.155807E+01
.179584E+01	.999655E+00	.143977E+01
.182390E+01	.999697E+00	.133652E+01
.185196E+01	.999732E+00	.124549E+01
.188002E+01	.999761E+00	.116454E+01
.190808E+01	.999785E+00	.109197E+01
.193614E+01	.999805E+00	.102646E+01
.196420E+01	.999822E+00	.966947E+00
.199226E+01	.999837E+00	.912566E+00
.202032E+01	.999850E+00	.862621E+00
.204838E+01	.999861E+00	.816522E+00

.207644E+01	.999871E+00	.773783E+00
.210450E+01	.999880E+00	.734003E+00
.213256E+01	.999887E+00	.696833E+00
.216062E+01	.999894E+00	.661978E+00
.218868E+01	.999900E+00	.629187E+00
.221674E+01	.999906E+00	.598235E+00
.224480E+01	.999911E+00	.568938E+00
.227286E+01	.999915E+00	.541135E+00
.230092E+01	.999919E+00	.514669E+00
.232898E+01	.999923E+00	.489423E+00
.235704E+01	.999926E+00	.465277E+00
.238510E+01	.999929E+00	.442138E+00
.241316E+01	.999932E+00	.419907E+00
.244122E+01	.999934E+00	.398514E+00
.246928E+01	.999936E+00	.377878E+00
.249734E+01	.999939E+00	.357937E+00
.252540E+01	.999940E+00	.338629E+00
.255346E+01	.999942E+00	.319905E+00
.258152E+01	.999944E+00	.301716E+00
.260958E+01	.999945E+00	.284016E+00
.263764E+01	.999947E+00	.266763E+00
.266570E+01	.999948E+00	.249920E+00
.269376E+01	.999949E+00	.233452E+00
.272182E+01	.999950E+00	.217326E+00
.274988E+01	.999951E+00	.201509E+00
.277794E+01	.999952E+00	.185980E+00

------------- END OF RUN, SYSTEM # 1 -------------

D. MAGNITUDE, PHASE OF THE AUGMENTED IIR NOTCH FILTER: P=2

INPUT DATA FOR SYSTEM # 1

INPUT DATA SOURCEFILE: digfreq.tst
DEGREE OF NUMERATOR = 4
DEGREE OF DENOMINATOR = 4
dsorce = F
NUMBER OF FREQUENCY POINTS = 100 MAGNITUDE OPTION = STD
STARTING VALUE OF THETA = .000000E+00
INCREMENT OF THETA = .280600E-01

THE NUMERATOR COEFFICIENTS b(0),b(1)...b(L) ARE:

.9900E+00 -.3299E-02 .1861E-01 .6141E+00 -.8625E+00

THE DENOMINATOR COEFFICIENTS c(0),c(1)...c(N) ARE:

.1000E+01 .0000E+00 .0000E+00 .6108E+00 -.8539E+00

OUTPUT DATA FOR SYSTEM # 1

THETA (RADIANS)	MAGNITUDE	PHASE (DEGREES)
.000000E+00	.999940E+00	.000000E+00
.280600E-01	.999940E+00	-.194012E-01
.561200E-01	.999940E+00	-.388423E-01
.841800E-01	.999940E+00	-.583613E-01
.112240E+00	.999939E+00	-.780022E-01
.140300E+00	.999938E+00	-.978035E-01
.168360E+00	.999938E+00	-.117802E+00
.196420E+00	.999937E+00	-.138056E+00
.224480E+00	.999936E+00	-.158606E+00
.252540E+00	.999934E+00	-.179500E+00
.280600E+00	.999933E+00	-.200794E+00
.308660E+00	.999932E+00	-.222545E+00
.336720E+00	.999930E+00	-.244805E+00
.364780E+00	.999928E+00	-.267638E+00
.392840E+00	.999926E+00	-.291136E+00
.420900E+00	.999923E+00	-.315351E+00
.448960E+00	.999920E+00	-.340386E+00
.477020E+00	.999917E+00	-.366321E+00
.505080E+00	.999914E+00	-.393270E+00
.533140E+00	.999910E+00	-.421345E+00
.561200E+00	.999906E+00	-.450679E+00
.589260E+00	.999901E+00	-.481422E+00
.617320E+00	.999896E+00	-.513734E+00
.645380E+00	.999891E+00	-.547815E+00
.673440E+00	.999884E+00	-.583885E+00
.701500E+00	.999876E+00	-.622197E+00
.729560E+00	.999868E+00	-.663044E+00
.757620E+00	.999858E+00	-.706773E+00
.785680E+00	.999847E+00	-.753796E+00
.813740E+00	.999835E+00	-.804595E+00
.841800E+00	.999820E+00	-.859751E+00
.869860E+00	.999803E+00	-.919967E+00
.897920E+00	.999783E+00	-.986102E+00
.925980E+00	.999759E+00	-.105921E+01
.954040E+00	.999731E+00	-.114062E+01
.982100E+00	.999697E+00	-.123199E+01
.101016E+01	.999655E+00	-.133548E+01
.103822E+01	.999603E+00	-.145388E+01
.106628E+01	.999537E+00	-.159094E+01
.109434E+01	.999453E+00	-.175175E+01
.112240E+01	.999343E+00	-.194343E+01
.115046E+01	.999193E+00	-.217625E+01
.117852E+01	.998985E+00	-.246558E+01
.120658E+01	.998681E+00	-.283557E+01
.123464E+01	.998213E+00	-.332630E+01

.126270E+01	.997440E+00	-.400966E+01
.129076E+01	.996024E+00	-.502849E+01
.131882E+01	.992994E+00	-.671234E+01
.134688E+01	.984530E+00	-.100258E+02
.137494E+01	.942856E+00	-.194057E+02
.140300E+01	.381887E-01	-.877335E+02
.143106E+01	.939924E+00	.200011E+02
.145912E+01	.984111E+00	.102594E+02
.148718E+01	.992867E+00	.687151E+01
.151524E+01	.995970E+00	.516108E+01
.154330E+01	.997413E+00	.412995E+01
.157136E+01	.998198E+00	.344001E+01
.159942E+01	.998671E+00	.294542E+01
.162748E+01	.998979E+00	.257306E+01
.165554E+01	.999189E+00	.228223E+01
.168360E+01	.999340E+00	.204849E+01
.171166E+01	.999452E+00	.185626E+01
.173972E+01	.999537E+00	.169518E+01
.176778E+01	.999603E+00	.155806E+01
.179584E+01	.999655E+00	.143976E+01
.182390E+01	.999697E+00	.133650E+01
.185196E+01	.999732E+00	.124548E+01
.188002E+01	.999761E+00	.116454E+01
.190808E+01	.999785E+00	.109196E+01
.193614E+01	.999805E+00	.102646E+01
.196420E+01	.999822E+00	.966945E+00
.199226E+01	.999837E+00	.912560E+00
.202032E+01	.999850E+00	.862616E+00
.204838E+01	.999861E+00	.816516E+00
.207644E+01	.999871E+00	.773779E+00
.210450E+01	.999880E+00	.734001E+00
.213256E+01	.999887E+00	.696832E+00
.216062E+01	.999894E+00	.661977E+00
.218868E+01	.999900E+00	.629182E+00
.221674E+01	.999906E+00	.598231E+00
.224480E+01	.999911E+00	.568938E+00
.227286E+01	.999915E+00	.541134E+00
.230092E+01	.999919E+00	.514669E+00
.232898E+01	.999923E+00	.489424E+00
.235704E+01	.999926E+00	.465279E+00
.238510E+01	.999929E+00	.442136E+00
.241316E+01	.999932E+00	.419910E+00
.244122E+01	.999934E+00	.398511E+00
.246928E+01	.999936E+00	.377878E+00
.249734E+01	.999938E+00	.357935E+00
.252540E+01	.999940E+00	.338632E+00
.255346E+01	.999942E+00	.319908E+00
.258152E+01	.999944E+00	.301717E+00
.260958E+01	.999945E+00	.284017E+00
.263764E+01	.999947E+00	.266763E+00
.266570E+01	.999948E+00	.249920E+00
.269376E+01	.999949E+00	.233450E+00
.272182E+01	.999950E+00	.217325E+00
.274988E+01	.999951E+00	.201511E+00
.277794E+01	.999952E+00	.185983E+00

------------ END OF RUN, SYSTEM # 1 ------------

E. MAGNITUDE, PHASE OF THE AUGMENTED IIR NOTCH FILTER: P=3

INPUT DATA FOR SYSTEM # 1

```
INPUT DATA SOURCEFILE: digfreq.tst
DEGREE OF NUMERATOR =   5
DEGREE OF DENOMINATOR =   5
dsorce = F
NUMBER OF FREQUENCY POINTS = 100   MAGNITUDE OPTION = STD
STARTING VALUE OF THETA =  .000000E+00
INCREMENT OF THETA =  .280600E-01

THE NUMERATOR COEFFICIENTS b(0),b(1)...b(L) ARE:

 .9900E+00   -.3299E-02    .1861E-01    .9374E-02   -.6610E+00
-.6047E+00



THE DENOMINATOR COEFFICIENTS c(0),c(1)...c(N) ARE:

 .1000E+01    .0000E+00    .0000E+00    .0000E+00   -.6524E+00
-.5987E+00
```

OUTPUT DATA FOR SYSTEM # 1

THETA (RADIANS)	MAGNITUDE	PHASE (DEGREES)
.000000E+00	.999937E+00	.000000E+00
.280600E-01	.999938E+00	-.194494E-01
.561200E-01	.999938E+00	-.388970E-01
.841800E-01	.999938E+00	-.584018E-01
.112240E+00	.999938E+00	-.780285E-01
.140300E+00	.999938E+00	-.978157E-01
.168360E+00	.999937E+00	-.117817E+00
.196420E+00	.999936E+00	-.138065E+00
.224480E+00	.999935E+00	-.158610E+00
.252540E+00	.999934E+00	-.179497E+00

.280600E+00	.999933E+00	-.200787E+00
.308660E+00	.999931E+00	-.222538E+00
.336720E+00	.999929E+00	-.244798E+00
.364780E+00	.999927E+00	-.267635E+00
.392840E+00	.999925E+00	-.291127E+00
.420900E+00	.999923E+00	-.315344E+00
.448960E+00	.999920E+00	-.340375E+00
.477020E+00	.999917E+00	-.366312E+00
.505080E+00	.999914E+00	-.393263E+00
.533140E+00	.999910E+00	-.421337E+00
.561200E+00	.999906E+00	-.450671E+00
.589260E+00	.999902E+00	-.481411E+00
.617320E+00	.999896E+00	-.513727E+00
.645380E+00	.999891E+00	-.547810E+00
.673440E+00	.999884E+00	-.583876E+00
.701500E+00	.999876E+00	-.622189E+00
.729560E+00	.999868E+00	-.663036E+00
.757620E+00	.999858E+00	-.706766E+00
.785680E+00	.999847E+00	-.753794E+00
.813740E+00	.999835E+00	-.804590E+00
.841800E+00	.999820E+00	-.859744E+00
.869860E+00	.999803E+00	-.919960E+00
.897920E+00	.999783E+00	-.986096E+00
.925980E+00	.999759E+00	-.105921E+01
.954040E+00	.999731E+00	-.114061E+01
.982100E+00	.999697E+00	-.123199E+01
.101016E+01	.999655E+00	-.133547E+01
.103822E+01	.999603E+00	-.145388E+01
.106628E+01	.999537E+00	-.159094E+01
.109434E+01	.999453E+00	-.175175E+01
.112240E+01	.999343E+00	-.194343E+01
.115046E+01	.999193E+00	-.217624E+01
.117852E+01	.998985E+00	-.246558E+01
.120658E+01	.998681E+00	-.283557E+01
.123464E+01	.998213E+00	-.332630E+01
.126270E+01	.997440E+00	-.400966E+01
.129076E+01	.996024E+00	-.502848E+01
.131882E+01	.992994E+00	-.671234E+01
.134688E+01	.984530E+00	-.100258E+02
.137494E+01	.942856E+00	-.194057E+02
.140300E+01	.381933E-01	-.877273E+02
.143106E+01	.939924E+00	.200011E+02
.145912E+01	.984110E+00	.102594E+02
.148718E+01	.992867E+00	.687151E+01
.151524E+01	.995970E+00	.516108E+01
.154330E+01	.997413E+00	.412995E+01
.157136E+01	.998198E+00	.344001E+01
.159942E+01	.998671E+00	.294542E+01
.162748E+01	.998979E+00	.257306E+01
.165554E+01	.999189E+00	.228223E+01
.168360E+01	.999340E+00	.204848E+01
.171166E+01	.999452E+00	.185626E+01
.173972E+01	.999537E+00	.169518E+01
.176778E+01	.999603E+00	.155805E+01
.179584E+01	.999655E+00	.143975E+01
.182390E+01	.999697E+00	.133650E+01

.185196E+01	.999732E+00	.124548E+01
.188002E+01	.999761E+00	.116453E+01
.190808E+01	.999785E+00	.109196E+01
.193614E+01	.999805E+00	.102645E+01
.196420E+01	.999822E+00	.966937E+00
.199226E+01	.999837E+00	.912553E+00
.202032E+01	.999850E+00	.862608E+00
.204838E+01	.999861E+00	.816508E+00
.207644E+01	.999871E+00	.773770E+00
.210450E+01	.999880E+00	.733993E+00
.213256E+01	.999887E+00	.696823E+00
.216062E+01	.999894E+00	.661968E+00
.218868E+01	.999900E+00	.629177E+00
.221674E+01	.999906E+00	.598225E+00
.224480E+01	.999911E+00	.568932E+00
.227286E+01	.999915E+00	.541125E+00
.230092E+01	.999919E+00	.514664E+00
.232898E+01	.999922E+00	.489416E+00
.235704E+01	.999926E+00	.465272E+00
.238510E+01	.999929E+00	.442129E+00
.241316E+01	.999932E+00	.419903E+00
.244122E+01	.999934E+00	.398506E+00
.246928E+01	.999936E+00	.377872E+00
.249734E+01	.999938E+00	.357931E+00
.252540E+01	.999940E+00	.338631E+00
.255346E+01	.999942E+00	.319911E+00
.258152E+01	.999943E+00	.301721E+00
.260958E+01	.999945E+00	.284018E+00
.263764E+01	.999946E+00	.266771E+00
.266570E+01	.999948E+00	.249936E+00
.269376E+01	.999949E+00	.233465E+00
.272182E+01	.999950E+00	.217347E+00
.274988E+01	.999951E+00	.201533E+00
.277794E+01	.999952E+00	.186002E+00

------------- END OF RUN, SYSTEM # 1 -------------

APPENDIX B. PROGRAM AND DATA OF DIFFERENCE EQUATION

A. PROGRAM FOR DIFFERENCE EQUATION CALCULATION

```
C                                DIFFEQ.FOR          VERSION: 2/03/88
C
C
C  PURPOSE:  THIS PROGRAM COMPUTES THE ITERATIVE SOLUTION TO A
C            LINEAR, TIME-INVARIANT (LTI) DIFFERENCE EQUATION.
C            THE DIFFERENCE EQUATION MUST BE IN THE FORM:
C            y(ns) = a(1)*y(ns-1) + ... + a(N)*y(ns-N) +
C                    b(0)*x(ns) + b(1)*x(ns-1) + ... + b(L)*x(ns-L).
C            THE PROGRAM CONSISTS OF A MAIN PROGRAM AND TWO
C            SUBROUTINES.  SUBROUTINE diffeq IS CALLED BY THE MAIN
C            PROGRAM TO ITERATIVELY SOLVE THE DIFFERENCE EQUATIONS AND
C            SUBROUTINE xgen ALLOWS THE USER THE OPTION OF GENERATING
C            THE INPUT SEQUENCE x() BY WRITING THE APPROPRIATE
C            EQUATIONS.  IF THE USER ELECTS TO GENERATE THE SEQUENCE
C            x() BY USING xgen THEN THE PROGRAM MUST BE COMPILED
C            AGAIN BEFORE EXECUTION.  THE USER HAS THE OPTION OF
C            SELECTING ONE OF TWO OPERATING MODES: BATCH OR TEST.
C            IN BATCH MODE THE AMOUNT OF INTERFACE WITH THE USER
C            IS MINIMIZED AND IT IS ASSUMED THAT THE INPUT DATA
C            HAS BEEN STORED IN THE DEFAULT FILE 'DIFFEQ.IN'.  IN
C            TEST MODE THE USER IS PROMPTED FOR THE NAME OF THE
C            INPUT FILE OR HAS THE OPTION OF PERFORMING A TEST RUN
C            USING THE INPUT DATA STORED IN THE FILE 'DIFFEQ.TST'.
C            IT IS RECOMMENDED THAT FIRST-TIME USERS SELECT THE
C            TEST MODE AND MAKE A TRIAL RUN WITH THE PRESTORED
C            INPUT DATA.  THE TEST MODE ECHOES PORTIONS OF THE
C            INPUT DATA ONTO THE MONITOR TO ALLOW VERIFICATION OF
C            ITS ACCURACY.  BOTH BATCH AND TEST MODES ALLOW THE
C            USER TO SOLVE UP TO FOUR DIFFERENCE EQUATIONS IN A
C            SINGLE PROGRAM EXECUTION.  THE OUTPUT OF THE PROGRAM
C            'DIFFEQ.FOR' IS STORED IN THE ARRAY y().  THE OUTPUT IS
C            STORED IN TABULAR FORM IN THE OUTPUT FILE 'DIFFEQ.FOR'
C            AND IN A FORM SUITABLE FOR PLOTTING IN THE FILE
C            'DIFFEQ.DAT'.
C
C
C*********************************** INPUT ***********************************
C
C
C  THIS PROGRAM ASSUMES THAT EACH DIFFERENCE EQUATION IS IN THE
C  CANONICAL FORM:
C
C  y(ns) = a(1)*y(ns-1) + ... + a(N)*y(ns-N) +
C          b(0)*x(ns) + b(1)*x(ns-1) + ... + b(L)*x(ns-L)
C
C  L = A NON-NEGATIVE INTEGER, THE NUMBER OF INPUT DELAYS.
C  N = A NON-NEGATIVE INTEGER, THE NUMBER OF OUTPUT DELAYS.
C
```

```
C  a(1)...a(N) = REAL COEFFICIENTS OF THE OUTPUT TERMS.
C  b(0)...b(L) = REAL COEFFICIENTS OF THE INPUT TERMS.
C
C  THE INPUT PARAMETERS SHOULD BE STORED IN A FILE NAMED
C  'DIFFEQ.IN'.  ALL OF THE READ STATEMENTS USED BY THIS PROGRAM
C  REQUIRE FORMATTED INPUT.  PARTICULAR ATTENTION SHOULD BE PAID
C  TO THE FORMATS, ESPECIALLY THE USE OF THE DECIMAL POINT TO
C  DENOTE 'REAL' NUMBERS.  THE INPUT PARAMETERS REQUIRED BY THE
C  PROGRAM ARE LISTED BELOW.
C
C
C  NAME         TYPE       RANGE (ARRAYS)          RESTRICTIONS
C  ----         ----       --------------        ----------------
C  numsys      INTEGER                           1 <= numsys <= 4
C  L           INTEGER                             0 <= L <= 128
C  N           INTEGER                             0 <= N <= 128
C  nstop       INTEGER                           0 <= nstop <= 999
C  xsorce      CHARACTER                            'F' OR 'S'
C  b(k)        REAL          0,1, ..., L           0 <= L <= 128
C  a(k)        REAL          1,2, ..., N           0 <= N <= 128
C  y(k)        REAL           -N, ..., -1          1 <= N <= 128
C  x(ns)       REAL        0,1, ..., nstop       0 <= nstop <= 999
C
C  WHERE:
C
C  numsys = THE NUMBER OF SYSTEMS TO BE EVALUATED.
C           THIS INTEGER VALUE MUST OCCUR AT THE TOP OF THE INPUT
C           FILE. IT DELINEATES THE NUMBER OF SYSTEMS TO BE READ BY
C           THE PROGRAM AND ANALYZED.  FOR EACH SYSTEM (1...numsys)
C           THE PARAMETERS BELOW MUST APPEAR IN THE INPUT FILE.
C
C  L = AN INTEGER VALUE THAT SPECIFIES THE MAXIMUM NUMBER OF DELAYS
C      IN THE INPUT SEQUENCE.
C
C  N = AN INTEGER VALUE THAT SPECIFIES THE MAXIMUM NUMBER OF DELAYS
C      IN THE OUTPUT SEQUENCE.
C
C  nstop = AN INTEGER VALUE THAT SPECIFIES THE LARGEST TIME INDEX
C          (ns) FOR WHICH THE DIFFERENCE EQUATION IS TO BE SOLVED.
C
C  xsorce = A CHARACTER VALUE OF 'F' OR 'S' DENOTING WHETHER THE
C           INPUT SEQUENCE x() IS TO BE READ FROM THE INPUT FILE (F)
C           OR TO BE GENERATED (S) USING THE SUBROUTINE xgen.  THIS
C           LATER OPTION IS ATTRACTIVE WHEN nstop IS A LARGE NUMBER
C           AND THE INPUT SEQUENCE x() CAN BE READILY DESCRIBED BY AN
C           ANALYTICAL EXPRESSION.  IF xsorce = 'S' THE USER MUST
C           PROVIDE THE APPROPRIATE FORTRAN STATEMENTS IN THE SPACE
C           PROVIDED IN SUBROUTINE xgen AND THE PROGRAM MUST BE
C           RECOMPILED BEFORE EXECUTION.
C
C  b(k) = REAL COEFFICIENTS OF THE INPUT SEQUENCE x(ns-k) IN THE
C         ORDER:  b(0), b(1), ..., b(L).
C
C  a(k) = REAL COEFFICIENTS OF THE OUTPUT SEQUENCE y(ns-k) IN THE
C         ORDER:  a(1), a(2), ..., a(N).  IF N = 0 THEN THE EQUATION
C         IS NON-RECURSIVE AND NO a(k) COEFFICIENTS SHOULD BE IN
```

```
C         THE INPUT FILE.
C
C  y(k) = THE INITIAL CONDITIONS FOR THE OUTPUT SEQUENCE IF THE
C         DIFFERENCE EQUATION IS RECURSIVE, I.E., N > 0.  THIS
C         PROGRAM CALCULATES THE SOLUTION TO THE DIFFERENCE
C         EQUATION FROM ns = 0 TO ns = nstop THEREFORE THE INITIAL
C         CONDITIONS y(-N) TO y(-1) MUST BE PROVIDED IN THE INPUT
C         FILE IN THE ORDER: y(-N), y(-N+1), ..., y(-1).  IF N = 0
C         THEN THE EQUATION IS NON-RECURSIVE AND NO INITIAL
C         CONDITIONS SHOULD BE GIVEN IN THE INPUT FILE.
C
C  x(ns) = THE INPUT SEQUENCE.  IF xsorce = 'F' THEN THE INPUT
C          SEQUENCE x(0), ..., x(nstop) MUST BE PROVIDED BY THE USER
C          IN THE INPUT FILE.  IF xsorce = 'S' THEN THE USER HAS
C          ELECTED TO GENERATE THE INPUT SEQUENCE BY PROVIDING THE
C          APPROPRIATE FORTRAN STATEMENTS IN THE SUBROUTINE xgen.
C
C  NOTE:  THE INPUT FORMAT STATEMENTS OCCUR IN THE MAIN PROGRAM
C         FOLLOWING THE CAPTION: *********  INPUT FORMAT  *********.
C         THE FORM OF THE INPUT DATA FILE IS:
C
C  LINE #                      ENTRIES                    FORMAT
C
C  ______                      _______                    ______
C    1                         numsys                       i1
C    2                    L,N,nstop,xsorce         i3,t11,i3,t21,i3,t31,a1
C  NEXT NB LINES          b(k), k=0,1,...,L              6f10.0
C  NEXT NA LINES           a(k), k=1,...,N               6f10.0
C  NEXT NY LINES          y(k), k= -N,...,-1             6f10.0
C  NEXT NX LINES        x(ns), ns= 0,...,nstop           6f10.0
C
C  WHERE: NB = 1 + (L/6 ROUNDED DOWN TO THE NEXT SMALLER INTEGER)
C
C         NA = 0 IF N = 0 OR
C         NA = 1 + ((N-1)/6 ROUNDED DOWN TO THE NEXT SMALLER INTEGER)
C
C         NY = 0 IF N = 0 OR
C         NY = 1 + ((N-1)/6 ROUNDED DOWN TO THE NEXT SMALLER INTEGER)
C
C         NX = 0 IF xsorce = 'S' OR
C         NX = 1 + (nstop/6 ROUNDED DOWN TO THE NEXT SMALLER INTEGER)
C              IF xsorce = 'F'
C
C  *NOTE:  FOR numsys > 1 THE FORMAT OF LINES 2... IS REPEATED.
C
C          THE FORMAT f10.0 USED FOR INPUT DATA PERMITS THE DECIMAL
C          POINT TO BE PLACED ANYWHERE IN THE FIELD OF 10 COLUMNS
C          AND ALSO ALLOWS THE EXPONENTIAL FORMAT TO BE USED (EG.
C          3146.2 = 3.1462E+03).
C
C
C********************************  OUTPUT  ********************************
C
C
C  THE INPUT DATA AS WELL AS THE OUTPUT DATA ARE STORED IN TABULAR
C  FORM IN THE FILE 'DIFFEQ.OUT'.  ADDITIONALLY, THE INPUT SEQUENCE
C  AND THE OUTPUT SEQUENCE ARE WRITTEN INTO THE FILE 'DIFFEQ.DAT' TO
```

```
C  FACILITATE PLOTTING BY A SEPARATE, USER SUPPLIED PROGRAM.  THE
C  FORMAT OF THE DATA IN 'DIFFEQ.DAT' IS: e12.6, 2X, e12.6. THE FIRST
C  ENTRY CORRESPONDS TO THE ORDINATE VALUE, AND THE SECOND ENTRY, THE
C  ABSCISSA VALUE.  ADDITIONAL HEADER INFORMATION IS WRITTEN INTO
C  'DIFFEQ.DAT' TO ALLOW FOR CONTROL AND LABELING OF EACH PLOT.
C
C
C******************************  EXAMPLE  ******************************
C
C
C  THE INPUT PARAMETERS FOR THE SYSTEM DESCRIBED BELOW ARE STORED IN
C  THE SAMPLE INPUT FILE 'DIFFEQ.TST' AND CAN BE USED FOR A TRIAL
C  RUN IN TEST MODE.
C
C  DIFFERENCE EQUATION:
C
C  y(ns) = 1.2*y(ns-1) + 1.5*x(ns)
C
C  GOAL:    TO OBTAIN THE SOLUTION TO THIS DIFFERENCE EQUATION FOR
C           ns = 0 TO ns = 10, GIVEN: x(0)...x(10) = 100.0 AND
C           THE INITIAL CONDITION y(-1) = 25.0.
C
C
C  THE INPUT FILE IS:
C
C  1
C  000       001       010       F
C  1.5
C  1.2
C  25.0
C  100.0     100.0     100.0     100.0     100.0     100.0
C  100.0     100.0     100.0     100.0     100.0
C
C
C  THE RESULTING OUTPUT FILE 'DIFFEQ.OUT' IS:
C
C                 INPUT DATA FOR PROBLEM #  1
C
C   PROBLEM # 1    INPUT DATA SOURCEFILE: DIFFEQ.TST
C   THE NUMBER OF INPUT DELAYS: L =   0
C   THE NUMBER OF OUTPUT DELAYS: N =   1
C   THE VALUE OF nstop IS:  10
C   THE COEFFICIENTS b(0), b(1), ..., b(L) ARE:
C
C  .150000E+01
C
C
C   THE COEFFICIENTS a(1), ..., a(N) ARE:
C
C  .120000E+01
C
C
C                OUTPUT DATA FOR PROBLEM #  1
C
C    ns        x(ns)              y(ns)
C    -1      .000000E+00        .250000E+02
```

```
C       0       .100000E+03         .180000E+03
C       1       .100000E+03         .366000E+03
C       2       .100000E+03         .589200E+03
C       3       .100000E+03         .857040E+03
C       4       .100000E+03         .117845E+04
C       5       .100000E+03         .156414E+04
C       6       .100000E+03         .202697E+04
C       7       .100000E+03         .258236E+04
C       8       .100000E+03         .324883E+04
C       9       .100000E+03         .404860E+04
C      10       .100000E+03         .500832E+04
C
C ---------------- END OF PROBLEM # 1 ----------------
C
C
C  FOR ILLUSTRATIVE PURPOSES THE INPUT SEQUENCE x() COULD HAVE BEEN
C  GENERATED BY SPECIFYING xsorce = 'S' AND WRITING THE APPROPRIATE
C  FORTRAN STATEMENTS INTO SUBROUTINE xgen.  THE STATEMENTS THAT
C  COULD BE USED TO ACCOMPLISH THIS ARE WRITTEN INTO THE SUBROUTINE
C  BUT ARE 'COMMENTED OUT'.
C
C
C****************************  MAIN PROGRAM  ****************************


      character infile*12, mode*1, xsorce*1
      real a(1:128), b(0:128), y(-128:850), x(-128:850), ii

C  PROMPT USER FOR MODE: BATCH OR TEST.

      write(*,1115)
      read(*,1117) mode
      if((mode.eq.'Y').or.(mode.eq.'y')) then
mode = 'Y'
write(*,1118)
read(*,1119) infile
      else
infile = 'DIFFEQ.IN'
      endif

C  UNIT=1 DEFINED AS INPUT FILE.  UNITS=2,3 DEFINED AS OUTPUT FILES.

      open(unit=1,file=infile,status='old',iostat=ierr,err=999)
      open(unit=2,file='DIFFEQ.OUT')
      open(unit=3,file='DIFFEQ.DAT')

C  READ INPUT PARAMETERS AND CONDUCT ERROR CHECKS.

      read(1,1000) numsys

      if((numsys.le.0).or.(numsys.gt.4)) then
write(*,1126) numsys
stop 'The allowed values for numsys are 1 <= numsys <= 4.'
      endif
      numplts = 2*numsys
      write(3,2000) numplts
```

```
      do 10 nprob=1, numsys

read(1,1001) L, N, nstop, xsorce

if((L.lt.0).or.(L.gt.128)) then
  write(*,1124) nprob, 'L', L
  stop 'The allowed values for ''L'' are:  0 <= L <= 128.'
    endif

if((N.lt.0).or.(N.gt.128)) then
  write(*,1124) nprob, 'N', N
  stop 'The allowed values for ''N'' are:  0 <= N <= 128.'
    endif

if((nstop.lt.0).or.(nstop.gt.850)) then
  write(*,1127) nprob, nstop
  stop 'The allowed values for nstop are: 0 <= nstop <= 850.'
    endif

if((xsorce.eq.'F').or.(xsorce.eq.'f')) then
  xsorce = 'F'
elseif((xsorce.eq.'S').or.(xsorce.eq.'s')) then
  xsorce = 'S'
     else
  write(*,1128) nprob, xsorce
  stop 'The allowed values for ''xsorce'' are: ''F'' or ''S''.'
    endif

C  INITIALIZE EACH ARRAY TO ZERO BEFORE EACH RUN.

data a/128*0.0/, b/129*0.0/
data y/979*0.0/, x/979*0.0/

C  READ THE COEFFICIENTS b(), a() AND THE INITIAL CONDITIONS
C  y(-N)...y(-1).

read(1,1002) (b(k), k=0,L)
if(N.gt.0) then
  icstart = -N
  read(1,1002) (a(k), k=1,N)
  read(1,1002) (y(k), k=icstart,-1)
    endif

C  FOR xsorce = 'F' READ THE ARRAY x() FROM THE INPUT FILE.
C  FOR xsorce = 'S' CALL xgen TO GENERATE THE ARRAY x().

if(xsorce.eq.'F') then
  read(1,1002) (x(k), k=0,nstop)
     else
  call xgen(x,nstop,nprob)
    endif

C  FOR TEST MODE ECHO INPUT PARAMETERS ONTO MONITOR (UNIT = *).

if(mode.eq.'Y') then
```

```
  write(*,1007)
  write(*,1120) nprob, infile
  write(*,1110) 'INPUT', 'L', L
  write(*,1110) 'OUTPUT', 'N', N
  write(*,1112) nstop
  write(*,1004)
  write(*,1005) (b(k),k=0,L)
  if(N.eq.0) then
    write(*,1131)
     else
    write(*,1006)
    write(*,1005) (a(k),k=1,N)
   endif
  write(*,1123) nprob
  pause 'END OF RUN, STRIKE <CR> WHEN READY TO CONTINUE.'
   endif

C  WRITE INPUT DATA INTO FILE: DIFFEQ.OUT.

write(2,1008) 'INPUT', nprob
write(2,1120) nprob, infile
write(2,1110) 'INPUT', 'L', L
write(2,1110) 'OUTPUT', 'N', N
write(2,1112) nstop
write(2,1004)
write(2,1005) (b(k),k=0,L)
if(N.eq.0) then
  write(2,1131)
    else
  write(2,1006)
  write(2,1005) (a(k),k=1,N)
   endif

C  WRITE THE INPUT SEQUENCE INTO FILE: DIFFEQ.DAT.

write(3,2001) nstop + 1
write(3,*) 'INPUT SEQUENCE x(ns)'
write(3,*) 'SAMPLE # (ns)'
write(3,*) 'x(ns)'
do 54 ns=0, nstop
  ii = ns
  write(3,2010) ii, x(ns)
54      continue

C  CALL diffeq TO COMPUTE THE SOLUTION TO THE DIFFERENCE EQUATION.

call diffeq(N,L,a,b,x,y,nstop)

C  WRITE RESULTS INTO FILE: DIFFEQ.DAT.

write(3,2001) N + nstop + 1
write(3,*) 'OUTPUT SEQUENCE Y(ns)'
write(3,*) 'SAMPLE # (ns)'
write(3,*) 'y(ns)'
do 55 ns= -N, nstop
  ii = ns
```

```
  write(3,2010) ii, y(ns)
55      continue

C  WRITE RESULTS INTO FILE: DIFFEQ.OUT.

write(2,1008) 'OUTPUT', nprob
write(2,1129)
do 102 ns= -N, nstop
  write(2,1130) ns, x(ns), y(ns)
102     continue
write(2,1123) nprob

10    continue

      write(*,1121)
999   close(unit=1)
      close(unit=2)
      close(unit=3)

      if(ierr.gt.0) then
write(*,1116) ierr
      endif

C*********  INPUT FORMAT  *********

1000  format(i1)
1001  format(i3,t11,i3,t21,i3,t31,a1)
1002  format(6f10.0)

C*************************************

1004  format(t4,'THE COEFFICIENTS b(0), b(1), ..., b(L) ARE:',/)
1005  format(6(1x,e12.6))
1006  format(//,t4,'THE COEFFICIENTS a(1), ..., a(N) ARE:',/)
1007  format(//////////)
1008  format(///,t16,a6,' DATA FOR PROBLEM # ',i1,//)
1110  format(t4,'THE NUMBER OF ',a6,' DELAYS: ',a1,' = ',i3)
1112  format(t4,'THE VALUE OF nstop IS: ',i3)
1115 0format(1x,'DO YOU WISH TO RUN THIS PROGRAM IN TEST',
     1' MODE ?    (Y/N) <CR> : ', ,)
1116 0format(///,1x,'ERROR OPENING INPUT FILE, PROGRAM TERMINATED.',
     1//,1x,'ERROR CODE:',i4,/////)
1117  format(a1)
1118 0format(/////,1x,'TYPE THE NAME OF YOUR DATA FILE FOLLOWED',
     1' BY <CR>.',/,' IF YOU DESIRE TO MAKE A TEST RUN USING THE',
     2' SAMPLE DATA ALREADY STORED',/,' IN THE FILE: DIFFEQ.TST',
     3 '        TYPE: DIFFEQ.TST <CR>',/,' FILENAME: ', ,)
1119  format(a12)
1120  format(///,t4,'PROBLEM # ',i1,'   INPUT DATA SOURCEFILE: ',a12)
1121 0format(//,' TABULAR OUTPUT DATA IS STORED IN FILE: DIFFEQ.OUT.'
     1,/,' PLOTTING DATA IS STORED IN FILE: DIFFEQ.DAT.')
1122  format(i3)
1123  format(/,1x,16('-'),' END OF PROBLEM #',i2,2x,16('-'),//)
1124 0format(//,' For problem #',i2,' the value for ',a1,' is: ',i3,
```

```
     1'.  This value is not allowed.')
1126  format(//,' numsys = ',i4,'.  This value is not allowed.')
1127 0format(//,' For problem #',i2,' the value for ''nstop'' is: ',
     1i3,'.  This value is not allowed.')
1128 0format(//,' For problem #',i2,' the value for ''xsorce'' is: ',
     1a1,'.  This value is not allowed.')
1129  format(t6,'ns',t16,'x(ns)',t35,'y(ns)')
1130  format(t4,i4,t11,e14.6,t30,e14.6)
1131  format(/,' THIS SYSTEM IS NON-RECURSIVE, I.E., N = 0.')
2000  format(i1)
2001  format(i3)
2010  format(e12.6,2x,e12.6)

      end


C                          SUBROUTINE: diffeq


C  PURPOSE:  THIS SUBROUTINE COMPUTES THE SOLUTION TO A DIFFERENCE
C            EQUATION.  ALL PARAMETERS DESCRIBING THE EQUATION, AND
C            THE INPUT AND OUTPUT SEQUENCES x() AND y() ARE PASSED
C            TO THE SUBROUTINE BY THE MAIN PROGRAM.

      subroutine diffeq(N,L,a,b,x,y,nstop)
      real x(-128:nstop), y(-128:nstop), a(1:N), b(0:L)

      do 500 ns=0, nstop
y(ns) = 0.0
do 501 k=0, max(N,L)
  y(ns) = y(ns) + a(k)*y(ns-k) + b(k)*x(ns-k)
501     continue
500   continue

      return
      end


C                          SUBROUTINE: xgen


C  PURPOSE:  THIS SUBROUTINE ALLOWS THE USER TO GENERATE VALUES FOR
C            THE ARRAY x().  IF xsorce = 'S' THE MAIN PROGRAM WILL
C            CALL THIS SUBROUTINE.  IF xsorce = 'F' THIS SUBROUTINE
C            WILL NOT BE CALLED BY THE MAIN PROGRAM.


      subroutine xgen(x,nstop,nprob)
      real x(-128:nstop)

C*********************************************************************************
C  DEVELOP THE ALGORITHM FOR GENERATING VALUES OF x() IN THIS SPACE.
C  THE STATEMENTS TYPED IN MUST FOLLOW STANDARD FORTRAN 77 RULES AND
C  MAY USE FORTRAN 77 INTRINSIC FUNCTIONS SUCH AS: SIN(), COS(), ...
C  NOTE THAT THE VALUE nprob CAN BE USED IN A LOGICAL 'IF' STATEMENT
C  TO MATCH THE GENERATING FUNCTIONS TO THE CORRESPONDING SYSTEM
```

```
C  EQUATION READ FROM THE INPUT FILE IF MORE THAN ONE SYSTEM OF
C  EQUATIONS EXIST.  AN EXAMPLE OF AN ALGORITHM GENERATING VALUES
C  FOR x() IS:
C
C
C***  EXAMPLE  ***
C
C     if(nprob.eq.1) then
C       do 1 k=0, nstop
C         x(k) = 100.0
C 1     continue
C     endif
      x(0) = 1.0
      do 1 k=1,nstop
 x(k) = 0.0
1     continue




C*******************************************************************************

      return
      end
```

B. INPUT, OUTPUT DATA OF SECOND ORDER IIR NOTCH FILTER: P=0

```
                INPUT DATA FOR PROBLEM # 1


   PROBLEM # 1   INPUT DATA SOURCEFILE: diffeq.tst
   THE NUMBER OF  INPUT DELAYS: L =   2
   THE NUMBER OF OUTPUT DELAYS: N =   2
   THE VALUE OF nstop IS: 400
   THE COEFFICIENTS b(0), b(1), ..., b(L) ARE:

  .990000E+00 -.329930E+00  .990000E+00


   THE COEFFICIENTS a(1), ..., a(N) ARE:
```

.329930E+00 -.980100E+00

OUTPUT DATA FOR PROBLEM # 1

ns	x(ns)	y(ns)
-2	.000000E+00	.000000E+00
-1	.000000E+00	.000000E+00
0	.100000E+01	.990000E+00
1	.000000E+00	-.329930E-02
2	.000000E+00	.186125E-01
3	.000000E+00	.937446E-02
4	.000000E+00	-.151492E-01
5	.000000E+00	-.141861E-01
6	.000000E+00	.101673E-01
7	.000000E+00	.172583E-01
8	.000000E+00	-.427095E-02
9	.000000E+00	-.183239E-01
10	.000000E+00	-.185966E-02
11	.000000E+00	.173457E-01
12	.000000E+00	.754553E-02
13	.000000E+00	-.145111E-01
14	.000000E+00	-.121830E-01
15	.000000E+00	.102028E-01
16	.000000E+00	.153068E-01
17	.000000E+00	-.494956E-02
18	.000000E+00	-.166352E-01
19	.000000E+00	-.637380E-03
20	.000000E+00	.160938E-01
21	.000000E+00	.593453E-02
22	.000000E+00	-.138156E-01
23	.000000E+00	-.103746E-01
24	.000000E+00	.101178E-01
25	.000000E+00	.135063E-01
26	.000000E+00	-.546028E-02
27	.000000E+00	-.150390E-01
28	.000000E+00	.389786E-03
29	.000000E+00	.148684E-01
30	.000000E+00	.452349E-02
31	.000000E+00	-.130801E-01
32	.000000E+00	-.874898E-02
33	.000000E+00	.993321E-02
34	.000000E+00	.118521E-01
35	.000000E+00	-.582516E-02
36	.000000E+00	-.135382E-01
37	.000000E+00	.124259E-02
38	.000000E+00	.136787E-01
39	.000000E+00	.329516E-02
40	.000000E+00	-.123194E-01
41	.000000E+00	-.729411E-02
42	.000000E+00	.966765E-02
43	.000000E+00	.103386E-01
44	.000000E+00	-.606425E-02
45	.000000E+00	-.121336E-01

46	.000000E+00	.194032E-02
47	.000000E+00	.125324E-01
48	.000000E+00	.223309E-02
49	.000000E+00	-.115462E-01
50	.000000E+00	-.599809E-02
51	.000000E+00	.933748E-02
52	.000000E+00	.895944E-02
53	.000000E+00	-.619567E-02
54	.000000E+00	-.108253E-01
55	.000000E+00	.250079E-02
56	.000000E+00	.114349E-01
57	.000000E+00	.132171E-02
58	.000000E+00	-.107713E-01
59	.000000E+00	-.484919E-02
60	.000000E+00	.895708E-02
61	.000000E+00	.770790E-02
62	.000000E+00	-.623577E-02
63	.000000E+00	-.961188E-02
64	.000000E+00	.294043E-02
65	.000000E+00	.103907E-01
66	.000000E+00	.546304E-03
67	.000000E+00	-.100037E-01
68	.000000E+00	-.383596E-02
69	.000000E+00	.853905E-02
70	.000000E+00	.657691E-02
71	.000000E+00	-.619920E-02
72	.000000E+00	-.849133E-02
73	.000000E+00	.327429E-02
74	.000000E+00	.940264E-02
75	.000000E+00	-.106918E-03
76	.000000E+00	-.925080E-02
77	.000000E+00	-.294733E-02
78	.000000E+00	.809430E-02
79	.000000E+00	.555923E-02
80	.000000E+00	-.609907E-02
81	.000000E+00	-.746086E-02
82	.000000E+00	.351613E-02
83	.000000E+00	.847247E-02
84	.000000E+00	-.650839E-03
85	.000000E+00	-.851860E-02
86	.000000E+00	-.217265E-02
87	.000000E+00	.763226E-02
88	.000000E+00	.464753E-02
89	.000000E+00	-.594701E-02
90	.000000E+00	-.651714E-02
91	.000000E+00	.367847E-02
92	.000000E+00	.760109E-02
93	.000000E+00	-.109744E-02
94	.000000E+00	-.781190E-02
95	.000000E+00	-.150178E-02
96	.000000E+00	.716096E-02
97	.000000E+00	.383451E-02
98	.000000E+00	-.575334E-02
99	.000000E+00	-.565640E-02
100	.000000E+00	.377263E-02
101	.000000E+00	.678855E-02

102	.000000E+00	-.145781E-02
103	.000000E+00	-.713443E-02
104	.000000E+00	-.925062E-03
105	.000000E+00	.668725E-02
106	.000000E+00	.311298E-02
107	.000000E+00	-.552711E-02
108	.000000E+00	-.487459E-02
109	.000000E+00	.380885E-02
110	.000000E+00	.603424E-02
111	.000000E+00	-.174217E-02
112	.000000E+00	-.648895E-02
113	.000000E+00	-.433395E-03
114	.000000E+00	.621683E-02
115	.000000E+00	.247589E-02
116	.000000E+00	-.527624E-02
117	.000000E+00	-.416741E-02
118	.000000E+00	.379629E-02
119	.000000E+00	.533699E-02
120	.000000E+00	-.195991E-02
121	.000000E+00	-.587742E-02
122	.000000E+00	-.182246E-04
123	.000000E+00	.575444E-02
124	.000000E+00	.191643E-02
125	.000000E+00	-.500764E-02
126	.000000E+00	-.353046E-02
127	.000000E+00	.374319E-02
128	.000000E+00	.469519E-02
129	.000000E+00	-.211961E-02
130	.000000E+00	-.530108E-02
131	.000000E+00	.328445E-03
132	.000000E+00	.530396E-02
133	.000000E+00	.142802E-02
134	.000000E+00	-.472726E-02
135	.000000E+00	-.295927E-02
136	.000000E+00	.365683E-02
137	.000000E+00	.410688E-02
138	.000000E+00	-.222908E-02
139	.000000E+00	-.476059E-02
140	.000000E+00	.614057E-03
141	.000000E+00	.486845E-02
142	.000000E+00	.100441E-02
143	.000000E+00	-.444019E-02
144	.000000E+00	-.244937E-02
145	.000000E+00	.354370E-02
146	.000000E+00	.356981E-02
147	.000000E+00	-.229540E-02
148	.000000E+00	-.425609E-02
149	.000000E+00	.845509E-03
150	.000000E+00	.445035E-02
151	.000000E+00	.639621E-03
152	.000000E+00	-.415076E-02
153	.000000E+00	-.199635E-02
154	.000000E+00	.340950E-02
155	.000000E+00	.308152E-02
156	.000000E+00	-.232497E-02
157	.000000E+00	-.378727E-02

158	.000000E+00	.102916E-02
159	.000000E+00	.405146E-02
160	.000000E+00	.328015E-03
161	.000000E+00	-.386261E-02
162	.000000E+00	-.159588E-02
163	.000000E+00	.325922E-02
164	.000000E+00	.263944E-02
165	.000000E+00	-.232353E-02
166	.000000E+00	-.335351E-02
167	.000000E+00	.117087E-02
168	.000000E+00	.367308E-02
169	.000000E+00	.642924E-04
170	.000000E+00	-.357878E-02
171	.000000E+00	-.124376E-02
172	.000000E+00	.309721E-02
173	.000000E+00	.224087E-02
174	.000000E+00	-.229624E-02
175	.000000E+00	-.295387E-02
176	.000000E+00	.127597E-02
177	.000000E+00	.331607E-02
178	.000000E+00	-.156510E-03
179	.000000E+00	-.330172E-02
180	.000000E+00	-.935942E-03
181	.000000E+00	.292722E-02
182	.000000E+00	.188310E-02
183	.000000E+00	-.224768E-02
184	.000000E+00	-.258720E-02
185	.000000E+00	.134936E-02
186	.000000E+00	.298091E-02
187	.000000E+00	-.339014E-03
188	.000000E+00	-.303344E-02
189	.000000E+00	-.668554E-03
190	.000000E+00	.275250E-02
191	.000000E+00	.156338E-02
192	.000000E+00	-.218192E-02
193	.000000E+00	-.225215E-02
194	.000000E+00	.139544E-02
195	.000000E+00	.266773E-02
196	.000000E+00	-.487510E-03
197	.000000E+00	-.277549E-02
198	.000000E+00	-.437908E-03
199	.000000E+00	.257578E-02
200	.000000E+00	.127902E-02
201	.000000E+00	-.210253E-02
202	.000000E+00	-.194725E-02
203	.000000E+00	.141823E-02
204	.000000E+00	.237642E-02
205	.000000E+00	-.605957E-03
206	.000000E+00	-.252905E-02
207	.000000E+00	-.240512E-03
208	.000000E+00	.239937E-02
209	.000000E+00	.102735E-02
210	.000000E+00	-.201267E-02
211	.000000E+00	-.167095E-02
212	.000000E+00	.142132E-02
213	.000000E+00	.210663E-02

214	.000000E+00	-.697998E-03
215	.000000E+00	-.229500E-02
216	.000000E+00	-.730823E-04
217	.000000E+00	.222522E-02
218	.000000E+00	.805795E-03
219	.000000E+00	-.191508E-02
220	.000000E+00	-.142160E-02
221	.000000E+00	.140794E-02
222	.000000E+00	.185783E-02
223	.000000E+00	-.766969E-03
224	.000000E+00	-.207391E-02
225	.000000E+00	.674610E-04
226	.000000E+00	.205490E-02
227	.000000E+00	.611853E-03
228	.000000E+00	-.181214E-02
229	.000000E+00	-.119756E-02
230	.000000E+00	.138096E-02
231	.000000E+00	.162935E-02
232	.000000E+00	-.815913E-03
233	.000000E+00	-.186612E-02
234	.000000E+00	.183989E-03
235	.000000E+00	.188968E-02
236	.000000E+00	.443136E-03
237	.000000E+00	-.170587E-02
238	.000000E+00	-.997137E-03
239	.000000E+00	.134294E-02
240	.000000E+00	.142037E-02
241	.000000E+00	-.847595E-03
242	.000000E+00	-.167175E-02
243	.000000E+00	.279167E-03
244	.000000E+00	.173059E-02
245	.000000E+00	.297362E-03
246	.000000E+00	-.159804E-02
247	.000000E+00	-.818687E-03
248	.000000E+00	.129613E-02
249	.000000E+00	.123003E-02
250	.000000E+00	-.864516E-03
251	.000000E+00	-.149078E-02
252	.000000E+00	.355459E-03
253	.000000E+00	.157839E-02
254	.000000E+00	.172373E-03
255	.000000E+00	-.149011E-02
256	.000000E+00	-.660574E-03
257	.000000E+00	.124251E-02
258	.000000E+00	.105737E-02
259	.000000E+00	-.868928E-03
260	.000000E+00	-.132301E-02
261	.000000E+00	.415134E-03
262	.000000E+00	.143365E-02
263	.000000E+00	.661320E-04
264	.000000E+00	-.138330E-02
265	.000000E+00	-.521209E-03
266	.000000E+00	.118381E-02
267	.000000E+00	.901413E-03
268	.000000E+00	-.862852E-03
269	.000000E+00	-.116816E-02

270	.000000E+00	.460272E-03
271	.000000E+00	.129677E-02
272	.000000E+00	-.232701E-04
273	.000000E+00	-.127864E-02
274	.000000E+00	-.399054E-03
275	.000000E+00	.112153E-02
276	.000000E+00	.761140E-03
277	.000000E+00	-.848092E-03
278	.000000E+00	-.102580E-02
279	.000000E+00	.492771E-03
280	.000000E+00	.116797E-02
281	.000000E+00	-.976162E-04
282	.000000E+00	-.117693E-02
283	.000000E+00	-.292632E-03
284	.000000E+00	.105697E-02
285	.000000E+00	.635534E-03
286	.000000E+00	-.826250E-03
287	.000000E+00	-.895491E-03
288	.000000E+00	.514358E-03
289	.000000E+00	.104737E-02
290	.000000E+00	-.158563E-03
291	.000000E+00	-.107885E-02
292	.000000E+00	-.200536E-03
293	.000000E+00	.991213E-03
294	.000000E+00	.523576E-03
295	.000000E+00	-.798744E-03
296	.000000E+00	-.776687E-03
297	.000000E+00	.526597E-03
298	.000000E+00	.934971E-03
299	.000000E+00	-.207643E-03
300	.000000E+00	-.984873E-03
301	.000000E+00	-.121428E-03
302	.000000E+00	.925211E-03
303	.000000E+00	.424267E-03
304	.000000E+00	-.766821E-03
305	.000000E+00	-.668821E-03
306	.000000E+00	.530897E-03
307	.000000E+00	.830670E-03
308	.000000E+00	-.246269E-03
309	.000000E+00	-.895391E-03
310	.000000E+00	-.540483E-04
311	.000000E+00	.859741E-03
312	.000000E+00	.336627E-03
313	.000000E+00	-.731569E-03
314	.000000E+00	-.571295E-03
315	.000000E+00	.528523E-03
316	.000000E+00	.734302E-03
317	.000000E+00	-.275737E-03
318	.000000E+00	-.810663E-03
319	.000000E+00	.278823E-05
320	.000000E+00	.795451E-03
321	.000000E+00	.259710E-03
322	.000000E+00	-.693935E-03
323	.000000E+00	-.483492E-03
324	.000000E+00	.520607E-03
325	.000000E+00	.645634E-03

326	.000000E+00	-.297233E-03
327	.000000E+00	-.730852E-03
328	.000000E+00	.501878E-04
329	.000000E+00	.732867E-03
330	.000000E+00	.192606E-03
331	.000000E+00	-.654736E-03
332	.000000E+00	-.404790E-03
333	.000000E+00	.508155E-03
334	.000000E+00	.564390E-03
335	.000000E+00	-.311833E-03
336	.000000E+00	-.656042E-03
337	.000000E+00	.891797E-04
338	.000000E+00	.672410E-03
339	.000000E+00	.134443E-03
340	.000000E+00	-.614672E-03
341	.000000E+00	-.334566E-03
342	.000000E+00	.492056E-03
343	.000000E+00	.490253E-03
344	.000000E+00	-.320515E-03
345	.000000E+00	-.586244E-03
346	.000000E+00	.120718E-03
347	.000000E+00	.614406E-03
348	.000000E+00	.843958E-04
349	.000000E+00	-.574335E-03
350	.000000E+00	-.272207E-03
351	.000000E+00	.473097E-03
352	.000000E+00	.422879E-03
353	.000000E+00	-.324162E-03
354	.000000E+00	-.521414E-03
355	.000000E+00	.145681E-03
356	.000000E+00	.559102E-03
357	.000000E+00	.416829E-04
358	.000000E+00	-.534224E-03
359	.000000E+00	-.217110E-03
360	.000000E+00	.451961E-03
361	.000000E+00	.361905E-03
362	.000000E+00	-.323564E-03
363	.000000E+00	-.461457E-03
364	.000000E+00	.164877E-03
365	.000000E+00	.506671E-03
366	.000000E+00	.557033E-05
367	.000000E+00	-.494751E-03
368	.000000E+00	-.168693E-03
369	.000000E+00	.429248E-03
370	.000000E+00	.306958E-03
371	.000000E+00	-.319432E-03
372	.000000E+00	-.406239E-03
373	.000000E+00	.179045E-03
374	.000000E+00	.457227E-03
375	.000000E+00	-.246287E-04
376	.000000E+00	-.456254E-03
377	.000000E+00	-.126393E-03
378	.000000E+00	.405474E-03
379	.000000E+00	.257656E-03
380	.000000E+00	-.312396E-03
381	.000000E+00	-.355598E-03

382	.000000E+00	.188857E-03
383	.000000E+00	.410831E-03
384	.000000E+00	-.495535E-04
385	.000000E+00	-.419005E-03
386	.000000E+00	-.896748E-04
387	.000000E+00	.381080E-03
388	.000000E+00	.213620E-03
389	.000000E+00	-.303017E-03
390	.000000E+00	-.309343E-03
391	.000000E+00	.194925E-03
392	.000000E+00	.367499E-03
393	.000000E+00	-.697972E-04
394	.000000E+00	-.383214E-03
395	.000000E+00	-.580256E-04
396	.000000E+00	.356444E-03
397	.000000E+00	.174472E-03
398	.000000E+00	-.291787E-03
399	.000000E+00	-.267270E-03
400	.000000E+00	.197800E-03

---------------- END OF PROBLEM # 1 ----------------

C. INPUT, OUTPUT DATA OF THE AUGMENTED IIR NOTCH FILTER: P = 1

INPUT DATA FOR PROBLEM # 1

PROBLEM # 1 INPUT DATA SOURCEFILE: diffeq.tst
THE NUMBER OF INPUT DELAYS: L = 3
THE NUMBER OF OUTPUT DELAYS: N = 3
THE VALUE OF nstop IS: 400
THE COEFFICIENTS b(0), b(1), ..., b(L) ARE:

.990000E+00 -.329900E-02 .881146E+00 .326631E+00

THE COEFFICIENTS a(1), ..., a(N) ARE:

.000000E+00 -.871246E+00 -.323364E+00

OUTPUT DATA FOR PROBLEM # 1

ns	x(ns)	y(ns)
-3	.000000E+00	.000000E+00
-2	.000000E+00	.000000E+00
-1	.000000E+00	.000000E+00
0	.100000E+01	.990000E+00
1	.000000E+00	-.329900E-02
2	.000000E+00	.186127E-01
3	.000000E+00	.937460E-02
4	.000000E+00	-.151494E-01
5	.000000E+00	-.141862E-01
6	.000000E+00	.101675E-01
7	.000000E+00	.172585E-01
8	.000000E+00	-.427105E-02
9	.000000E+00	-.183242E-01
10	.000000E+00	-.185964E-02
11	.000000E+00	.173460E-01
12	.000000E+00	.754559E-02
13	.000000E+00	-.145113E-01
14	.000000E+00	-.121831E-01
15	.000000E+00	.102029E-01
16	.000000E+00	.153069E-01
17	.000000E+00	-.494966E-02
18	.000000E+00	-.166353E-01
19	.000000E+00	-.637331E-03
20	.000000E+00	.160940E-01
21	.000000E+00	.593455E-02
22	.000000E+00	-.138158E-01
23	.000000E+00	-.103747E-01
24	.000000E+00	.101179E-01
25	.000000E+00	.135064E-01
26	.000000E+00	-.546039E-02
27	.000000E+00	-.150392E-01
28	.000000E+00	.389857E-03
29	.000000E+00	.148685E-01
30	.000000E+00	.452347E-02
31	.000000E+00	-.130802E-01
32	.000000E+00	-.874900E-02
33	.000000E+00	.993335E-02
34	.000000E+00	.118522E-01
35	.000000E+00	-.582528E-02
36	.000000E+00	-.135383E-01
37	.000000E+00	.124268E-02
38	.000000E+00	.136788E-01
39	.000000E+00	.329511E-02
40	.000000E+00	-.123195E-01
41	.000000E+00	-.729410E-02
42	.000000E+00	.966777E-02
43	.000000E+00	.103386E-01
44	.000000E+00	-.606436E-02
45	.000000E+00	-.121337E-01
46	.000000E+00	.194041E-02
47	.000000E+00	.125324E-01
48	.000000E+00	.223302E-02
49	.000000E+00	-.115463E-01
50	.000000E+00	-.599805E-02

51	.000000E+00	.933758E-02
52	.000000E+00	.895943E-02
53	.000000E+00	-.619577E-02
54	.000000E+00	-.108253E-01
55	.000000E+00	.250088E-02
56	.000000E+00	.114350E-01
57	.000000E+00	.132163E-02
58	.000000E+00	-.107714E-01
59	.000000E+00	-.484913E-02
60	.000000E+00	.895716E-02
61	.000000E+00	.770786E-02
62	.000000E+00	-.623586E-02
63	.000000E+00	-.961187E-02
64	.000000E+00	.294052E-02
65	.000000E+00	.103908E-01
66	.000000E+00	.546215E-03
67	.000000E+00	-.100038E-01
68	.000000E+00	-.383588E-02
69	.000000E+00	.853911E-02
70	.000000E+00	.657685E-02
71	.000000E+00	-.619928E-02
72	.000000E+00	-.849130E-02
73	.000000E+00	.327438E-02
74	.000000E+00	.940263E-02
75	.000000E+00	-.107009E-03
76	.000000E+00	-.925082E-02
77	.000000E+00	-.294724E-02
78	.000000E+00	.809434E-02
79	.000000E+00	.555915E-02
80	.000000E+00	-.609913E-02
81	.000000E+00	-.746081E-02
82	.000000E+00	.351621E-02
83	.000000E+00	.847244E-02
84	.000000E+00	-.650931E-03
85	.000000E+00	-.851860E-02
86	.000000E+00	-.217256E-02
87	.000000E+00	.763228E-02
88	.000000E+00	.464744E-02
89	.000000E+00	-.594707E-02
90	.000000E+00	-.651707E-02
91	.000000E+00	.367854E-02
92	.000000E+00	.760104E-02
93	.000000E+00	-.109753E-02
94	.000000E+00	-.781188E-02
95	.000000E+00	-.150168E-02
96	.000000E+00	.716097E-02
97	.000000E+00	.383442E-02
98	.000000E+00	-.575338E-02
99	.000000E+00	-.565632E-02
100	.000000E+00	.377269E-02
101	.000000E+00	.678848E-02
102	.000000E+00	-.145789E-02
103	.000000E+00	-.713439E-02
104	.000000E+00	-.924967E-03
105	.000000E+00	.668724E-02
106	.000000E+00	.311288E-02

107	.000000E+00	-.552713E-02
108	.000000E+00	-.487450E-02
109	.000000E+00	.380890E-02
110	.000000E+00	.603416E-02
111	.000000E+00	-.174225E-02
112	.000000E+00	-.648890E-02
113	.000000E+00	-.433302E-03
114	.000000E+00	.621681E-02
115	.000000E+00	.247579E-02
116	.000000E+00	-.527625E-02
117	.000000E+00	-.416731E-02
118	.000000E+00	.379633E-02
119	.000000E+00	.533691E-02
120	.000000E+00	-.195998E-02
121	.000000E+00	-.587736E-02
122	.000000E+00	-.181364E-04
123	.000000E+00	.575441E-02
124	.000000E+00	.191633E-02
125	.000000E+00	-.500764E-02
126	.000000E+00	-.353036E-02
127	.000000E+00	.374322E-02
128	.000000E+00	.469510E-02
129	.000000E+00	-.211967E-02
130	.000000E+00	-.530101E-02
131	.000000E+00	.328528E-03
132	.000000E+00	.530391E-02
133	.000000E+00	.142793E-02
134	.000000E+00	-.472725E-02
135	.000000E+00	-.295917E-02
136	.000000E+00	.365685E-02
137	.000000E+00	.410679E-02
138	.000000E+00	-.222913E-02
139	.000000E+00	-.476052E-02
140	.000000E+00	.614133E-03
141	.000000E+00	.486840E-02
142	.000000E+00	.100432E-02
143	.000000E+00	-.444016E-02
144	.000000E+00	-.244927E-02
145	.000000E+00	.354371E-02
146	.000000E+00	.356971E-02
147	.000000E+00	-.229544E-02
148	.000000E+00	-.425600E-02
149	.000000E+00	.845577E-03
150	.000000E+00	.445029E-02
151	.000000E+00	.639532E-03
152	.000000E+00	-.415072E-02
153	.000000E+00	-.199625E-02
154	.000000E+00	.340950E-02
155	.000000E+00	.308142E-02
156	.000000E+00	-.232500E-02
157	.000000E+00	-.378719E-02
158	.000000E+00	.102922E-02
159	.000000E+00	.405139E-02
160	.000000E+00	.327932E-03
161	.000000E+00	-.386257E-02
162	.000000E+00	-.159578E-02

163	.000000E+00	.325921E-02
164	.000000E+00	.263934E-02
165	.000000E+00	-.232355E-02
166	.000000E+00	-.335342E-02
167	.000000E+00	.117092E-02
168	.000000E+00	.367301E-02
169	.000000E+00	.642165E-04
170	.000000E+00	-.357873E-02
171	.000000E+00	-.124367E-02
172	.000000E+00	.309719E-02
173	.000000E+00	.224077E-02
174	.000000E+00	-.229626E-02
175	.000000E+00	-.295378E-02
176	.000000E+00	.127602E-02
177	.000000E+00	.331600E-02
178	.000000E+00	-.156579E-03
179	.000000E+00	-.330167E-02
180	.000000E+00	-.935856E-03
181	.000000E+00	.292720E-02
182	.000000E+00	.188300E-02
183	.000000E+00	-.224769E-02
184	.000000E+00	-.258711E-02
185	.000000E+00	.134939E-02
186	.000000E+00	.298083E-02
187	.000000E+00	-.339076E-03
188	.000000E+00	-.303338E-02
189	.000000E+00	-.668474E-03
190	.000000E+00	.275246E-02
191	.000000E+00	.156329E-02
192	.000000E+00	-.218191E-02
193	.000000E+00	-.225206E-02
194	.000000E+00	.139547E-02
195	.000000E+00	.266765E-02
196	.000000E+00	-.487564E-03
197	.000000E+00	-.277542E-02
198	.000000E+00	-.437834E-03
199	.000000E+00	.257574E-02
200	.000000E+00	.127893E-02
201	.000000E+00	-.210252E-02
202	.000000E+00	-.194717E-02
203	.000000E+00	.141825E-02
204	.000000E+00	.237634E-02
205	.000000E+00	-.606003E-03
206	.000000E+00	-.252899E-02
207	.000000E+00	-.240445E-03
208	.000000E+00	.239933E-02
209	.000000E+00	.102727E-02
210	.000000E+00	-.201266E-02
211	.000000E+00	-.167086E-02
212	.000000E+00	.142134E-02
213	.000000E+00	.210655E-02
214	.000000E+00	-.698037E-03
215	.000000E+00	-.229493E-02
216	.000000E+00	-.730215E-04
217	.000000E+00	.222517E-02
218	.000000E+00	.805719E-03

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220	.000000E+00	-.142152E-02
221	.000000E+00	.140795E-02
222	.000000E+00	.185775E-02
223	.000000E+00	-.767001E-03
224	.000000E+00	-.207384E-02
225	.000000E+00	.675152E-04
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232	.000000E+00	-.815938E-03
233	.000000E+00	-.186605E-02
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235	.000000E+00	.188963E-02
236	.000000E+00	.443071E-03
237	.000000E+00	-.170584E-02
238	.000000E+00	-.997063E-03
239	.000000E+00	.134294E-02
240	.000000E+00	.142030E-02
241	.000000E+00	-.847613E-03
242	.000000E+00	-.167168E-02
243	.000000E+00	.279207E-03
244	.000000E+00	.173054E-02
245	.000000E+00	.297304E-03
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247	.000000E+00	-.818618E-03
248	.000000E+00	.129612E-02
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251	.000000E+00	-.149071E-02
252	.000000E+00	.355493E-03
253	.000000E+00	.157833E-02
254	.000000E+00	.172321E-03
255	.000000E+00	-.149007E-02
256	.000000E+00	-.660510E-03
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260	.000000E+00	-.132295E-02
261	.000000E+00	.415163E-03
262	.000000E+00	.143360E-02
263	.000000E+00	.660854E-04
264	.000000E+00	-.138326E-02
265	.000000E+00	-.521150E-03
266	.000000E+00	.118379E-02
267	.000000E+00	.901347E-03
268	.000000E+00	-.862854E-03
269	.000000E+00	-.116809E-02
270	.000000E+00	.460295E-03
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273	.000000E+00	-.127860E-02
274	.000000E+00	-.399000E-03

275	.000000E+00	.112151E-02
276	.000000E+00	.761079E-03
277	.000000E+00	-.848089E-03
278	.000000E+00	-.102574E-02
279	.000000E+00	.492789E-03
280	.000000E+00	.116792E-02
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282	.000000E+00	-.117689E-02
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284	.000000E+00	.105694E-02
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305	.000000E+00	-.668769E-03
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307	.000000E+00	.830620E-03
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315	.000000E+00	.528524E-03
316	.000000E+00	.734253E-03
317	.000000E+00	-.275754E-03
318	.000000E+00	-.810621E-03
319	.000000E+00	.281864E-05
320	.000000E+00	.795419E-03
321	.000000E+00	.259670E-03
322	.000000E+00	-.693917E-03
323	.000000E+00	-.483446E-03
324	.000000E+00	.520605E-03
325	.000000E+00	.645588E-03
326	.000000E+00	-.297246E-03
327	.000000E+00	-.730811E-03
328	.000000E+00	.502140E-04
329	.000000E+00	.732835E-03
330	.000000E+00	.192569E-03

331	.000000E+00	-.654717E-03
332	.000000E+00	-.404747E-03
333	.000000E+00	.508149E-03
334	.000000E+00	.564346E-03
335	.000000E+00	-.311842E-03
336	.000000E+00	-.656002E-03
337	.000000E+00	.892021E-04
338	.000000E+00	.672377E-03
339	.000000E+00	.134410E-03
340	.000000E+00	-.614651E-03
341	.000000E+00	-.334527E-03
342	.000000E+00	.492049E-03
343	.000000E+00	.490211E-03
344	.000000E+00	-.320521E-03
345	.000000E+00	-.586206E-03
346	.000000E+00	.120736E-03
347	.000000E+00	.614374E-03
348	.000000E+00	.843669E-04
349	.000000E+00	-.574313E-03
350	.000000E+00	-.272171E-03
351	.000000E+00	.473087E-03
352	.000000E+00	.422840E-03
353	.000000E+00	-.324164E-03
354	.000000E+00	-.521377E-03
355	.000000E+00	.145696E-03
356	.000000E+00	.559070E-03
357	.000000E+00	.416575E-04
358	.000000E+00	-.534201E-03
359	.000000E+00	-.217077E-03
360	.000000E+00	.451950E-03
361	.000000E+00	.361869E-03
362	.000000E+00	-.323564E-03
363	.000000E+00	-.461421E-03
364	.000000E+00	.164889E-03
365	.000000E+00	.506640E-03
366	.000000E+00	.554827E-05
367	.000000E+00	-.494727E-03
368	.000000E+00	-.168663E-03
369	.000000E+00	.429235E-03
370	.000000E+00	.306924E-03
371	.000000E+00	-.319430E-03
372	.000000E+00	-.406206E-03
373	.000000E+00	.179054E-03
374	.000000E+00	.457197E-03
375	.000000E+00	-.246476E-04
376	.000000E+00	-.456231E-03
377	.000000E+00	-.126367E-03
378	.000000E+00	.405459E-03
379	.000000E+00	.257625E-03
380	.000000E+00	-.312392E-03
381	.000000E+00	-.355566E-03
382	.000000E+00	.188864E-03
383	.000000E+00	.410802E-03
384	.000000E+00	-.495696E-04
385	.000000E+00	-.418981E-03
386	.000000E+00	-.896512E-04

387	.000000E+00	.381065E-03
388	.000000E+00	.213592E-03
389	.000000E+00	-.303011E-03
390	.000000E+00	-.309313E-03
391	.000000E+00	.194929E-03
392	.000000E+00	.367471E-03
393	.000000E+00	-.698106E-04
394	.000000E+00	-.383191E-03
395	.000000E+00	-.580047E-04
396	.000000E+00	.356428E-03
397	.000000E+00	.174446E-03
398	.000000E+00	-.291779E-03
399	.000000E+00	-.267242E-03
400	.000000E+00	.197802E-03

---------------- END OF PROBLEM # 1 ----------------

D. INPUT, OUTPUT DATA OF THE AUGMENTED IIR NOTCH FILTER: P=2

INPUT DATA FOR PROBLEM # 1

PROBLEM # 1 INPUT DATA SOURCEFILE: diffeq.tst
THE NUMBER OF INPUT DELAYS: L = 4
THE NUMBER OF OUTPUT DELAYS: N = 4
THE VALUE OF nstop IS: 400
THE COEFFICIENTS b(0), b(1), ..., b(L) ARE:

.990000E+00 -.329900E-02 .186124E-01 .614081E+00 -.862533E+00

THE COEFFICIENTS a(1), ..., a(N) ARE:

.000000E+00 .000000E+00 -.610814E+00 .853908E+00

OUTPUT DATA FOR PROBLEM # 1

ns	x(ns)	y(ns)
-4	.000000E+00	.000000E+00
-3	.000000E+00	.000000E+00
-2	.000000E+00	.000000E+00

-1	.000000E+00	.000000E+00
0	.100000E+01	.990000E+00
1	.000000E+00	-.329900E-02
2	.000000E+00	.186124E-01
3	.000000E+00	.937506E-02
4	.000000E+00	-.151490E-01
5	.000000E+00	-.141858E-01
6	.000000E+00	.101669E-01
7	.000000E+00	.172586E-01
8	.000000E+00	-.427097E-02
9	.000000E+00	-.183234E-01
10	.000000E+00	-.186026E-02
11	.000000E+00	.173461E-01
12	.000000E+00	.754517E-02
13	.000000E+00	-.145102E-01
14	.000000E+00	-.121837E-01
15	.000000E+00	.102032E-01
16	.000000E+00	.153059E-01
17	.000000E+00	-.494841E-02
18	.000000E+00	-.166360E-01
19	.000000E+00	-.636442E-03
20	.000000E+00	.160924E-01
21	.000000E+00	.593604E-02
22	.000000E+00	-.138169E-01
23	.000000E+00	-.103729E-01
24	.000000E+00	.101156E-01
25	.000000E+00	.135084E-01
26	.000000E+00	-.546244E-02
27	.000000E+00	-.150363E-01
28	.000000E+00	.386692E-03
29	.000000E+00	.148715E-01
30	.000000E+00	.451996E-02
31	.000000E+00	-.130758E-01
32	.000000E+00	-.875350E-02
33	.000000E+00	.993800E-02
34	.000000E+00	.118465E-01
35	.000000E+00	-.581878E-02
36	.000000E+00	-.135450E-01
37	.000000E+00	.125013E-02
38	.000000E+00	.136700E-01
39	.000000E+00	.330475E-02
40	.000000E+00	-.123297E-01
41	.000000E+00	-.728235E-02
42	.000000E+00	.965436E-02
43	.000000E+00	.103531E-01
44	.000000E+00	-.608030E-02
45	.000000E+00	-.121155E-01
46	.000000E+00	.192010E-02
47	.000000E+00	.125545E-01
48	.000000E+00	.220828E-02
49	.000000E+00	-.115183E-01
50	.000000E+00	-.602890E-02
51	.000000E+00	.937158E-02
52	.000000E+00	.892123E-02
53	.000000E+00	-.615305E-02
54	.000000E+00	-.108724E-01

55	.000000E+00	.255325E-02
56	.000000E+00	.113763E-01
57	.000000E+00	.138688E-02
58	.000000E+00	-.108436E-01
59	.000000E+00	-.476854E-02
60	.000000E+00	.886717E-02
61	.000000E+00	.780770E-02
62	.000000E+00	-.634675E-02
63	.000000E+00	-.948809E-02
64	.000000E+00	.280269E-02
65	.000000E+00	.105437E-01
66	.000000E+00	.375915E-03
67	.000000E+00	-.981388E-02
68	.000000E+00	-.404702E-02
69	.000000E+00	.877377E-02
70	.000000E+00	.631545E-02
71	.000000E+00	-.590817E-02
72	.000000E+00	-.881493E-02
73	.000000E+00	.363443E-02
74	.000000E+00	.900161E-02
75	.000000E+00	.339247E-03
76	.000000E+00	-.974710E-02
77	.000000E+00	-.239484E-02
78	.000000E+00	.747933E-02
79	.000000E+00	.624335E-02
80	.000000E+00	-.686032E-02
81	.000000E+00	-.661345E-02
82	.000000E+00	.257313E-02
83	.000000E+00	.952163E-02
84	.000000E+00	-.181849E-02
85	.000000E+00	-.721898E-02
86	.000000E+00	-.361872E-02
87	.000000E+00	.924135E-02
88	.000000E+00	.285663E-02
89	.000000E+00	-.395398E-02
90	.000000E+00	-.873480E-02
91	.000000E+00	.614640E-02
92	.000000E+00	.485444E-02
93	.000000E+00	.195900E-02
94	.000000E+00	-.112130E-01
95	.000000E+00	.228329E-02
96	.000000E+00	.294866E-02
97	.000000E+00	.852188E-02
98	.000000E+00	-.109696E-01
99	.000000E+00	.148640E-03
100	.000000E+00	-.268740E-02
101	.000000E+00	.139773E-01
102	.000000E+00	-.945779E-02
103	.000000E+00	.176843E-02
104	.000000E+00	-.108323E-01
105	.000000E+00	.177122E-01
106	.000000E+00	-.915626E-02
107	.000000E+00	.812659E-02
108	.000000E+00	-.200687E-01
109	.000000E+00	.207174E-01
110	.000000E+00	-.127824E-01

111	.000000E+00	.191976E-01
112	.000000E+00	-.297913E-01
113	.000000E+00	.254984E-01
114	.000000E+00	-.226412E-01
115	.000000E+00	.345899E-01
116	.000000E+00	-.410138E-01
117	.000000E+00	.356029E-01
118	.000000E+00	-.404615E-01
119	.000000E+00	.545884E-01
120	.000000E+00	-.567688E-01
121	.000000E+00	.551160E-01
122	.000000E+00	-.678938E-01
123	.000000E+00	.812886E-01
124	.000000E+00	-.821409E-01
125	.000000E+00	.885345E-01
126	.000000E+00	-.107627E+00
127	.000000E+00	.119586E+00
128	.000000E+00	-.124219E+00
129	.000000E+00	.141341E+00
130	.000000E+00	-.164948E+00
131	.000000E+00	.177990E+00
132	.000000E+00	-.192404E+00
133	.000000E+00	.221445E+00
134	.000000E+00	-.249570E+00
135	.000000E+00	.269510E+00
136	.000000E+00	-.299557E+00
137	.000000E+00	.341534E+00
138	.000000E+00	-.377730E+00
139	.000000E+00	.413111E+00
140	.000000E+00	-.464408E+00
141	.000000E+00	.522361E+00
142	.000000E+00	-.574880E+00
143	.000000E+00	.636425E+00
144	.000000E+00	-.715627E+00
145	.000000E+00	.797194E+00
146	.000000E+00	-.879632E+00
147	.000000E+00	.980564E+00
148	.000000E+00	-.109802E+01
149	.000000E+00	.121802E+01
150	.000000E+00	-.135007E+01
151	.000000E+00	.150800E+01
152	.000000E+00	-.168159E+01
153	.000000E+00	.186472E+01
154	.000000E+00	-.207394E+01
155	.000000E+00	.231483E+01
156	.000000E+00	-.257492E+01
157	.000000E+00	.285909E+01
158	.000000E+00	-.318488E+01
159	.000000E+00	.354945E+01
160	.000000E+00	-.394512E+01
161	.000000E+00	.438677E+01
162	.000000E+00	-.488765E+01
163	.000000E+00	.544063E+01
164	.000000E+00	-.604826E+01
165	.000000E+00	.673134E+01
166	.000000E+00	-.749681E+01

167	.000000E+00	.834016E+01
168	.00000[illegible]E+00	-.927626E+01
169	.00000[illegible]E+00	.103271E+02
170	.000000E+00	-.114959E+02
171	.000000E+00	.127878E+02
172	.000000E+00	-.142290E+02
173	.000000E+00	.158402E+02
174	.000000E+00	-.176274E+02
175	.000000E+00	.196109E+02
176	.000000E+00	-.218257E+02
177	.000000E+00	.242932E+02
178	.000000E+00	-.270308E+02
179	.000000E+00	.300773E+02
180	.000000E+00	-.334757E+02
181	.000000E+00	.372549E+02
182	.000000E+00	-.414534E+02
183	.000000E+00	.461307E+02
184	.000000E+00	-.513410E+02
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187	.000000E+00	.707512E+02
188	.000000E+00	-.787379E+02
189	.000000E+00	.876183E+02
190	.000000E+00	-.975028E+02
191	.000000E+00	.108509E+03
192	.000000E+00	-.120753E+03
193	.000000E+00	.134374E+03
194	.000000E+00	-.149537E+03
195	.000000E+00	.166415E+03
196	.000000E+00	-.185190E+03
197	.000000E+00	.206083E+03
198	.000000E+00	-.229340E+03
199	.000000E+00	.255219E+03
200	.000000E+00	-.284013E+03
201	.000000E+00	.316059E+03
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203	.000000E+00	.391413E+03
204	.000000E+00	-.435575E+03
205	.000000E+00	.484725E+03
206	.000000E+00	-.539423E+03
207	.000000E+00	.600286E+03
208	.000000E+00	-.668018E+03
209	.000000E+00	.743398E+03
210	.000000E+00	-.827281E+03
211	.000000E+00	.920624E+03
212	.000000E+00	-.102450E+04
213	.000000E+00	.114011E+04
214	.000000E+00	-.126875E+04
215	.000000E+00	.141191E+04
216	.000000E+00	-.157123E+04
217	.000000E+00	.174852E+04
218	.000000E+00	-.194581E+04
219	.000000E+00	.216537E+04
220	.000000E+00	-.240970E+04
221	.000000E+00	.268160E+04
222	.000000E+00	-.298418E+04

223	.000000E+00	.332090E+04
224	.000000E+00	-.369562E+04
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226	.000000E+00	-.457667E+04
227	.000000E+00	.509308E+04
228	.000000E+00	-.566777E+04
229	.000000E+00	.630729E+04
230	.000000E+00	-.701898E+04
231	.000000E+00	.781098E+04
232	.000000E+00	-.869233E+04
233	.000000E+00	.967314E+04
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239	.000000E+00	.183719E+05
240	.000000E+00	-.204449E+05
241	.000000E+00	.227518E+05
242	.000000E+00	-.253191E+05
243	.000000E+00	.281760E+05
244	.000000E+00	-.313552E+05
245	.000000E+00	.348932E+05
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247	.000000E+00	.432119E+05
248	.000000E+00	-.480877E+05
249	.000000E+00	.535138E+05
250	.000000E+00	-.595520E+05
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252	.000000E+00	-.737494E+05
253	.000000E+00	.820710E+05
254	.000000E+00	-.913316E+05
255	.000000E+00	.101637E+06
256	.000000E+00	-.113105E+06
257	.000000E+00	.125868E+06
258	.000000E+00	-.140070E+06
259	.000000E+00	.155875E+06
260	.000000E+00	-.173463E+06
261	.000000E+00	.193036E+06
262	.000000E+00	-.214818E+06
263	.000000E+00	.239057E+06
264	.000000E+00	-.266031E+06
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266	.000000E+00	-.329454E+06
267	.000000E+00	.366628E+06
268	.000000E+00	-.407997E+06
269	.000000E+00	.454033E+06
270	.000000E+00	-.505265E+06
271	.000000E+00	.562277E+06
272	.000000E+00	-.625722E+06
273	.000000E+00	.696325E+06
274	.000000E+00	-.774896E+06
275	.000000E+00	.862332E+06
276	.000000E+00	-.959634E+06
277	.000000E+00	.106792E+07
278	.000000E+00	-.118841E+07

279	.000000E+00	.132251E+07
280	.000000E+00	-.147174E+07
281	.000000E+00	.163780E+07
282	.000000E+00	-.182260E+07
283	.000000E+00	.202826E+07
284	.000000E+00	-.225712E+07
285	.000000E+00	.251180E+07
286	.000000E+00	-.279523E+07
287	.000000E+00	.311063E+07
288	.000000E+00	-.346162E+07
289	.000000E+00	.385221E+07
290	.000000E+00	-.428688E+07
291	.000000E+00	.477059E+07
292	.000000E+00	-.530889E+07
293	.000000E+00	.590792E+07
294	.000000E+00	-.657455E+07
295	.000000E+00	.731639E+07
296	.000000E+00	-.814194E+07
297	.000000E+00	.906065E+07
298	.000000E+00	-.100830E+08
299	.000000E+00	.112207E+08
300	.000000E+00	-.124868E+08
301	.000000E+00	.138958E+08
302	.000000E+00	-.154637E+08
303	.000000E+00	.172086E+08
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307	.000000E+00	.263919E+08
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310	.000000E+00	-.363717E+08
311	.000000E+00	.404757E+08
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313	.000000E+00	.501253E+08
314	.000000E+00	-.557812E+08
315	.000000E+00	.620754E+08
316	.000000E+00	-.690797E+08
317	.000000E+00	.768744E+08
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319	.000000E+00	.952015E+08
320	.000000E+00	-.105944E+09
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326	.000000E+00	-.201216E+09
327	.000000E+00	.223920E+09
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330	.000000E+00	-.308593E+09
331	.000000E+00	.343413E+09
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339	.000000E+00	.807730E+09
340	.000000E+00	-.898871E+09
341	.000000E+00	.100030E+10
342	.000000E+00	-.111316E+10
343	.000000E+00	.123877E+10
344	.000000E+00	-.137855E+10
345	.000000E+00	.153410E+10
346	.000000E+00	-.170720E+10
347	.000000E+00	.189983E+10
348	.000000E+00	-.211420E+10
349	.000000E+00	.235276E+10
350	.000000E+00	-.261823E+10
351	.000000E+00	.291366E+10
352	.000000E+00	-.324243E+10
353	.000000E+00	.360829E+10
354	.000000E+00	-.401544E+10
355	.000000E+00	.446852E+10
356	.000000E+00	-.497273E+10
357	.000000E+00	.553384E+10
358	.000000E+00	-.615825E+10
359	.000000E+00	.685312E+10
360	.000000E+00	-.762640E+10
361	.000000E+00	.848693E+10
362	.000000E+00	-.944456E+10
363	.000000E+00	.105102E+11
364	.000000E+00	-.116962E+11
365	.000000E+00	.130159E+11
366	.000000E+00	-.144846E+11
367	.000000E+00	.161190E+11
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372	.000000E+00	-.275102E+11
373	.000000E+00	.306143E+11
374	.000000E+00	-.340687E+11
375	.000000E+00	.379129E+11
376	.000000E+00	-.421908E+11
377	.000000E+00	.469514E+11
378	.000000E+00	-.522492E+11
379	.000000E+00	.581448E+11
380	.000000E+00	-.647056E+11
381	.000000E+00	.720067E+11
382	.000000E+00	-.801317E+11
383	.000000E+00	.891734E+11
384	.000000E+00	-.992354E+11
385	.000000E+00	.110433E+12
386	.000000E+00	-.122893E+12
387	.000000E+00	.136760E+12
388	.000000E+00	-.152192E+12
389	.000000E+00	.169364E+12
390	.000000E+00	-.188475E+12

391	.000000E+00	.209741E+12
392	.000000E+00	-.233408E+12
393	.000000E+00	.259745E+12
394	.000000E+00	-.289053E+12
395	.000000E+00	.321669E+12
396	.000000E+00	-.357964E+12
397	.000000E+00	.398356E+12
398	.000000E+00	-.443305E+12
399	.000000E+00	.493325E+12
400	.000000E+00	-.548990E+12

---------------- END OF PROBLEM # 1 ----------------

E. INPUT, OUTPUT DATA OF THE AUGMENTED IIR NOTCH FILTER: P=3

INPUT DATA FOR PROBLEM # 1

PROBLEM # 1 INPUT DATA SOURCEFILE: diffeq.tst
THE NUMBER OF INPUT DELAYS: L = 5
THE NUMBER OF OUTPUT DELAYS: N = 5
THE VALUE OF nstop IS: 400
THE COEFFICIENTS b(0), b(1), ..., b(L) ARE:

.990000E+00 -.329900E-02 .186124E-01 .937440E-02 -.661007E+00 -.604706E+00

THE COEFFICIENTS a(1), ..., a(N) ARE:

.000000E+00 .000000E+00 .000000E+00 .652382E+00 .598659E+00

OUTPUT DATA FOR PROBLEM # 1

ns	x(ns)	y(ns)
-5	.000000E+00	.000000E+00
-4	.000000E+00	.000000E+00
-3	.000000E+00	.000000E+00
-2	.000000E+00	.000000E+00
-1	.000000E+00	.000000E+00
0	.100000E+01	.990000E+00
1	.000000E+00	-.329900E-02

2	.000000E+00	.186124E-01
3	.000000E+00	.937440E-02
4	.000000E+00	-.151485E-01
5	.000000E+00	-.141854E-01
6	.000000E+00	.101674E-01
7	.000000E+00	.172582E-01
8	.000000E+00	-.427053E-02
9	.000000E+00	-.183231E-01
10	.000000E+00	-.185917E-02
11	.000000E+00	.173458E-01
12	.000000E+00	.754575E-02
13	.000000E+00	-.145103E-01
14	.000000E+00	-.121822E-01
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400	.000000E+00	.123775E+03

---------------- END OF PROBLEM # 1 ----------------

LIST OF REFERENCES

1. Loomis, Herschel H. Jr. and Sinha, Bhaskar "High-speed Recursive Digital Filter Realization," *Circuits, Systems and Signal Processing, Vol. 3, No. 3, pp. 267-294, 1984.*

2. Gnanasekaran, R. "High Throughput 1-D State Space Digital Filter Implementation," *submitted for publication, IEEE Transactions on Communication.*

3. Rowe, C. L. "Detection and Analysis of Direct Sequence Spread Spectrum Signals," *MSEE Thesis, Naval Postgraduate School, Monterey, CA, pp. 68-73, September 1987.*

4. Lee. E. A., Lu. H. H. and Messerschmitt, D. G. "Fast Recursive Filtering with Multiple Slow Processing Elements," *IEEE Transaction Circuits Systems, Vol. CAS-32, pp. 1119-1129, November 1985.*

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3 11, Bangvak, Pasichareon
Bangkok
Thailand

Thesis
J2833 Jangsri
c.1 Infinite impulse response notch filter.